

ON SOME PROPERTIES OF THE STARS OF SPECTRAL TYPE **F**

BY

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LIC. PHIL. HB.

WITH 4 FIGURES

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INTRODUCTION.

1. The present paper is a statistical investigation of some properties of the *F*-stars. These stars have not been treated separately before except, as far as I know, in four memoirs, namely those of GYLLENBEEG *, EDDINGTON and HARTLEY **, PLUMMER ***, and CHARLIER and GYLLENBERG ****, other investigations including also stars of other spectral types. However, as pointed out by CHARLIER †, it is in certain respects desirable to have the stars of different spectral types examined separately and, the *O*-, *B*-, and *A*-stars †† having already been treated at the Lund Observatory, I adopted the *F*-stars as the object of my investigation.

2. The main part of this research is devoted to the determination of the frequency function of velocities, and I have made this determination with the ellipsoidal hypothesis as a starting point. The methods employed are given by CHARLIER †††, GYLLENBERG ††††, and WICKSELL *† in several important papers. However, according to the relatively small number of *F*-stars available to me, I have only computed the moments of the first and second orders, the higher moments being too badly determined. Confining myself to *F*-stars fainter than the sixth magnitude — of which practically all are known as regards apparent magnitude and proper motion — I had in all 560 stars. As, according to CHARLIER *††, it is generally not worth while computing the higher characteristics of a frequency distribution unless the population in question contains about 1000 individuals, my neglect of computing the dissymmetry and excess may thus seem justified. The theory and numerical results of this part of the paper are contained in chapter II.

The contents of the first chapter are partly the same as those of my monograph *L.M. I*, 77. It deals with the computation of the constant *R* introduced by

* Meddelanden från Lunds Astronomiska Observatorium, Serie II, n:o 13 (briefly *L.M. II*, 13).

** Monthly Notices, vol. 75 (briefly *M.N.* 75).

*** *M.N.* 76.

**** Meddelanden från Lunds Astronomiska Observatorium, Serie I, n:o 91 (briefly *L.M. I*, 91).

† *L.M. I*, 81.

†† *L.M. I*, 75; *L.M. II*, 14; *L.M. I*, 76.

††† *L.M. II*, 9.

†††† *L.M. II*, 13.

*† *L.M. II*, 12.

*†† CHARLIER: Grunddragen av den matematiska statistiken, p. 68.

CHARLIER* and used by him in determining the frequency function of velocities and the distribution in space of the *B*-stars. Further I have made a provisory determination of the number of *F*-stars per unit of volume in the neighbourhood of the sun.

Chapter III deals with those stars of spectral type *F* for which the parallax is determined. The correlation coefficients between the absolute magnitude on one side and one or both of the properties apparent magnitude and proper motion on the other side, are computed. An analytical dissection of the frequency curve of the absolute magnitude into two component curves is made, and by a method, given by CHARLIER and not yet published, I have made an approximative determination of the star density in the vicinity of the sun.

In the last chapter I have given a method for computing the stars' distribution in space when the frequency function of the apparent magnitude is known. Its numerical application requires the knowledge of the numerical value of the constant *R*, spoken of above and determined in chapter I. I have, however, on reasons given in chapter IV, not been able to apply this method on the *F*-stars, the material as yet available being too scanty, but I hope that it will come to use, when these stars are completely known up to about the tenth magnitude.

3. The data which I have used for the investigations in chapters I and II are

- 1) the apparent magnitude, *m*, taken from *Harvard Annals*, vol. 50 (briefly *H. A.* 50),
- 2) the spectral type, taken from the same source,
- 3) the proper motion components and
- 4) the equatorial coordinates, α and δ , taken from Boss's *Preliminary General Catalogue*,
- 5) the radial velocities taken from the catalogue of GYLLENBERG** completed by radial velocity determinations found in *Astrophysical Journal*, vol. 42 (briefly *Ap. J.* 42), and in other scattered papers.

4. The units for length and time used here are siriometer and stellar year respectively. Their definitions are:

- 1 siriometer (Sir.) = 10^6 times the mean distance from the earth to the sun;
- 1 stellar year (St.) = 10^6 tropical years.

* L.M. II, 14.

** L.M. II, 13.

CHAPTER I.

Determination of the constant R .

5. The investigation made in this chapter is based on the hypothesis that the absolute magnitude, M , for stars of a certain spectral type is approximately constant, this approximation being greater or smaller according to the type. Greatest is it for the B - and A -stars, these stars having a rather small dispersion in absolute magnitude, as has been shown by CHARLIER in *L.M. I*, 68. He there gives a diagram showing that the mean of the absolute magnitude and the square of its dispersion are connected by a linear relation. From my examination of the parallax stars of type F made in chapter III of this paper it is found that these stars have an absolute magnitude of about $+2^m$. By means of the above mentioned diagram of CHARLIER we can thus estimate the dispersion in the absolute magnitude to be probably not greater than about $1^{m.5}$ for the F -stars. This value is in fact also obtained in chapter III. On account of this relatively small dispersion I accepted the hypothesis of the constant absolute magnitude also for the F -stars and made a numerical determination of R , this quantity according to the relation (2) beneath also being assumed to be constant.

6. The meaning and definition of this constant and the method of its computation were given by CHARLIER in a preliminary state in *Astronomische Nachrichten* No 4801, and were later further developed in *L.M. II*, 14. In *L.M. I*, 77 I have reproduced it, and therefore I here only remark that R has the following forms

$$R = \rho \sqrt{\frac{f(T)}{k}},$$

where ρ is the radius of a star, T its temperature and k a constant, and

$$R = 10^{-0.2M}, \quad (1)$$

where M is the absolute magnitude of the star. From (1) we get

$$M = -5 \log R, \quad (2)$$

and the relation between the distance, r , and apparent magnitude, m , of a star is

$$r = R \cdot 10^{0.2m}. \quad (3)$$

From this equation we see that R means the distance at which a star would have an apparent magnitude equal to zero.

The proper motion components of a star being

$$u = \Delta\alpha \cos \delta, \quad v = \Delta\delta,$$

we have the linear cross motion components

$$U = ru, \quad V = rv.$$

Inserting here (3) it is found

$$U = Ru', \quad V = Rv', \quad (4)$$

where

$$u' = u \cdot 10^{0.2m}, \quad v' = v \cdot 10^{0.2m},$$

are called the reduced proper motion components. For the rest I refer to the two papers cited above.

7. The material available for this investigation contained in all 560 F -stars distributed in the four subclasses $F0$, $F2$, $F5$, and $F8$ as shown in the following table I.

TABLE I. Number of stars of the spectral type F .

Magn.	$F0$	$F2$	$F5$	$F8$	F
$\leq 4^{m.99}$	68.(0)	7.(0)	63.(0)	34.(1)	172.(1)
$5^{m.00}-5^{m.99}$	230.(1)	40.(1)	74.(0)	44.(2)	388.(4)
$\leq 5^{m.99}$	298.(1)	47.(1)	137.(0)	78.(3)	560.(5)

Before a numerical determination of the constant R could be made, it proved necessary to exclude certain stars having too great values of u' or v' . When determining which stars were to be rejected I endeavoured to retain as many as possible. In accordance with this it appeared that I could not make use of the rule which says that those individuals of a statistical population are to be excluded whose values of the argument exceed three times the dispersion of the proper argument, for in that case the number of stars available to me would have been diminished in a considerable degree. I therefore made provisory diagrams showing for each subclass the number of stars having u' resp. v' between certain limits and excluded those stars which by an inspection of these diagrams seemed to have too great values of the respective argument. Afterwards it turned out that the stars thus rejected had values of u' or v' exceeding about ± 3.5 times the dispersion of these arguments, and taking into consideration the relatively great excess which, as is well known, is to be found in the frequency curves of u' and v' , these limits seem to be justified. The number of the stars thus rejected are inserted in brackets in the table I above.

In order to get R we have use for the velocity, Ω_0 , of the sun. This quantity I have taken from GYLLENBERGS * investigation of 237 radial velocity stars of spectral type F . It has the value

$$\Omega_0 = 4,110 \pm 0,421 \text{ Sir./St.}$$

In the following table II I have collected the results found.

TABLE II. Values of R obtained from stars of spectral type F .

Spectr. class	Magn.	R	$\pm \varepsilon(R)$	M	A	D	N
$F0$ - stars	$\leq 4^m,99$	0,95	0,24	+ 0,10	82°,2	— 7°,0	68
	$5^m,00-5^m,99$	0,66	0,13	+ 0,90	92,3	— 34,4	229
	$\leq 5^m,99$	0,73	0,13	+ 0,69	89,3	— 30,1	297
$F2$ - stars	$\leq 4^m,99$	—	—	—	—	—	7
	$5^m,00-5^m,99$	0,49	0,16	+ 1,55	78,0	— 47,9	39
	$\leq 5^m,99$	0,57	0,19	+ 1,24	75,5	— 48,1	46
$F5$ - stars	$\leq 4^m,99$	0,68	0,16	+ 0,82	82,3	— 39,2	63
	$5^m,00-5^m,99$	0,56	0,13	+ 1,23	95,7	— 11,5	74
	$\leq 5^m,99$	0,64	0,12	+ 0,97	90,7	— 24,3	137
$F8$ - stars	$\leq 4^m,99$	0,75	0,46	+ 0,63	72,0	— 50,3	33
	$5^m,00-5^m,99$	0,59	0,27	+ 1,16	97,2	— 36,6	42
	$\leq 5^m,99$	0,66	0,25	+ 0,91	88,6	— 43,7	75
All F - stars	$\leq 4^m,99$	0,87	0,18	+ 0,30	76,2	— 31,7	171
	$5^m,00-5^m,99$	0,63	0,10	+ 1,01	91,9	— 32,0	384
	$\leq 5^m,99$	0,69	0,10	+ 0,81	88,4	— 32,0	555

M is the absolute magnitude, N the number of the stars, A and D the right ascension and declination respectively of the solar antapex. As to their determination I refer to *L.M. I*, 77. The mean error of R is computed by means of the formulæ page 6 in this paper.

8. Table III above shows that the quantity R as computed from all F -stars brighter than the sixth magnitude has a value of about 0,7 Sir. with a mean error amounting to about 0,1 Sir. The corresponding value of M is + 0^m,8.

Considering R as a function of the spectral type it is seen from the table III that R has its greatest value for the $F0$ -stars and its least value for the $F2$ -stars, whereas it begins to increase with the $F5$ -stars. This fact is stated both for all

* *L.M. II*, 13.

stars brighter than the sixth magnitude as well as for all stars brighter than the fifth magnitude and for stars of the fifth magnitude. Comparing this result with those obtained from similar investigations of stars of the spectral types *B*, *A*, *G*, *K*, *M*, and *N* made respectively by CHARLIER *, MALMQUIST **, FÄNGE ***, myself ***, FÄNGE ***, and GYLLENBERG ***, it is found that, plotting the values of *R* as ordinates against the subtypes of the spectral classes *B*, *A*, *F*, *G*, *K*, *M*, *N* as abscissæ — thereby observing that the order of the *B*-subtypes are according to CHARLIER † *B* 1, *B* 2, *B* 3, *B* 5, *B* 0, *B* 8, *B* 9 — we obtain a parabolalike curve which has its minimum somewhere about the *F* 2-stars. This would indicate that the *F*-stars should have the smallest intrinsic luminosity of all stars and that both *B*- and *A*-stars as well as *G*-, *K*-, *M*-, and *N*-stars should be absolutely brighter. This result conforms with those yielded by investigations of the mean parallax of different spectral types †† but is opposite to that obtained from parallax stars of all types †††, and the divergence can be explained by the hypothesis of giant- and dwarf-stars. According to this hypothesis the division into these two kinds of stars is not at hand in the case of the *B*-stars, but, proceeding in the order *A*, *F*, *G*, *K*, *M*, *N*, it becomes more and more perceptible. As now the redness of the stars also increases in the order above mentioned it is obvious that, confining ourselves to such low magnitudes as 5.0 or 6.0 we include of the reddest stars — i. e. *M*- and *N*-stars — chiefly such of great intrinsic luminosity, viz. the giants. The same is the case with stars of types *G* and *K*, though not in so high a degree as concerning the *M*- and *N*-stars.

From the table III is further to be seen that the apparently brighter stars give throughout a greater value of *R* than the apparently fainter ones. This may be due to the fact that *R* is not exactly constant, the frequency function of it (or of *M*) having a certain dispersion, from which results that the value of *R* for a certain group of stars is not the same as for another group. However, this explains only *why* *R* fluctuates from the one group mentioned above to the other, but not why it varies in the manner shown by table III, which variation should indicate a close positive correlation between apparent and absolute magnitude. This correlation I have not been able to explain. It should be produced if the proportion of dwarf stars were increased with increasing distance from us. However, the phenomenon of decreasing value of *R* for increasing apparent magnitude of the stars is also found in the case of the *B*- and *A*-stars, where the division into giants and dwarfs is supposed not to exist, and moreover the reason of it mentioned above does not conform with the facts found from double-stars that the masses of the stars are all of the same order of magnitude.

* L.M. II, 14.

** L.M. I, 76.

*** Not published.

† L.M. II, 14, page 25.

†† EDDINGTON: Stellar movements and the structure of the universe, chap. VIII.

††† Ibid. chap. III.

9. The numerical value of R now being known I computed by means of (3) the distance to every star. In this computation I used the four values of R obtained for every subtype containing stars fainter than the sixth magnitude. It was then possible to deduce an approximative value of the number of F -stars per cubicsirionometer in the vicinity of the sun. The subtype $F2$ having the least value of R , 0.57, we conclude that within a sphere having a radius equal to

$$r_0 = 0.57 \cdot 10^{0.2 \cdot 5.99} = 8.99 \text{ Sir.}$$

all subtypes are certainly represented. Knowing from *Harvard Annals*, vol. 50 and the computed distances that this sphere contains 374 F -stars, the number of F -stars per cubicsirionometer in our neighbourhood is obtained from

$$D_0 = \frac{n}{\frac{4}{3} \pi r_0^3}$$

by inserting $n = 374$, $r_0 = 8.99$. It is found that

$$D_0 = 0.124 \text{ } F\text{-stars/cub.-sir.}$$

This value of the star density is of course very approximative. In chapter III I have got an other value of D_0 in a wholly different manner but this latter value is somewhat larger than that here obtained.

10. To test the reliability of the computed star distances I have compared the parallaxes of the F -stars contained in *Groningen Publications N:o 24* with the computed parallaxes for the same stars. The latter are obtained by means of the formula

$$\pi = \frac{0''.206265}{r}, \quad (5)$$

where the unit of r is 1 Sir., which corresponds to a parallax equal to $0''.206265$. This comparison is given in table IV of *L.M. I*, 77, which also contains the velocity components U , V , and W of some stars. U and V are computed by means of (4), and W is directly observed.

By an inspection of this table it is seen that about 60 per cent of the computed parallaxes agree with the observed ones when the mean errors of the latter computed from the probable ones given in *Groningen Publications N:o 24*, are taken into account. The means of the observed and computed parallaxes are $0''.073$ and $0''.057$ respectively. Hence the observed parallaxes are on an average greater than the computed ones. This was, however, to be expected for the reason that those stars for which the parallaxes are determined have either great apparent luminosity or large proper motion, and both these properties indicate a relatively small distance. The greatest deviations between observed and computed parallax being found in the case of the stars ζ Tucanæ, η Cassiopeiæ, Polaris, 41 H. Andromedæ, 107 Piscium, Canopus, and 70 Virginis, we can state something about certain of them concerning their dimensions by the aid of the velocity components U , V , W . For ζ Tucanæ

the computed values of U and W are considerably greater than the observed value of W . As, probably, all three components, U , V , W ought to be of the same order of magnitude, it seems as if in this case we had used too great a value of R when computing U and W . Consequently this star should be a dwarf star. The same can to a lower degree of certainty be concluded about η Cassiopejæ and 41 H. Andromedæ. Looking at the values of U , V , W for Polaris and Canopus the contrary seems probable, and these stars should consequently be giant stars.

11. An earlier attempt to determine the parallaxes of the F -stars was made by PLUMMER*. His working hypothesis is quite another than ours. It consists in his assuming the peculiar motions of the stars to be parallel to the galactic plane. Knowing from previous investigations that the velocity surface is an ellipsoid somewhat flat in the plane of the galaxy this hypothesis is — like the one used by me — not strictly true. But from a comparison which PLUMMER makes between his computed parallaxes and the available observed ones, it seems as if the hypothesis of a constant absolute magnitude for the F -stars were to be preferred as a working hypothesis to that used by PLUMMER. For his comparison shows that only 40 per cent of the computed parallaxes agree with the observed ones when the mean errors in the observations are considered, whereas from my comparison the corresponding result was 60 per cent. Further it is obvious from the fundamental equation in PLUMMER's paper that in certain cases the computed parallaxes may have negative signs, and in fact he obtained negative parallaxes for one third of all his stars. This circumstance is wholly avoided by using equations (3) and (5).

As PLUMMER does not assume the absolute magnitude for the F -stars to be constant, it is possible for him to compute this quantity for every star. His mean results are:

Type	M_P	M_{Ch}
$F0$	+ 6, ^m ₄	— 0, ^m ₂
$F2$	+ 7, ^m ₆	+ 1, ^m ₀
$F5$	+ 7, ^m ₆	+ 1, ^m ₀
$F8$	+ 8, ^m ₄	+ 1, ^m ₈ .

Here M_P and M_{Ch} are the absolute magnitudes obtained by using PLUMMER's and CHARLIER's units of length which correspond to parallaxes equal to 0'',01 and 0'',206265 respectively. It is easy to deduce

$$M_{Ch} = M_P - 6^m,8.$$

The numbers in the small table above show that, as far as may be judged from PLUMMER's computed positive parallaxes, the absolute magnitude of the F -stars is nearly constant, which is a — may be weak — support for the working hypothesis used by me in this chapter. The mean of the absolute magnitude for the F -stars obtained by PLUMMER is about + 0^m,8, which agrees with the value of the same quantity obtained by me for F -stars fainter than the sixth magnitude and found in table II.

* M. N. 76.

CHAPTER II.

The frequency function of velocities.

12. In order to determine the frequency function of velocities I have, as mentioned in the introduction, accepted the ellipsoidal hypothesis, assuming the velocity surface to be an ellipsoid with three unequal axes. For this reason I shall in the sequel call it simply the velocity ellipsoid. I have determined it in four manners:

- A. From the theoretical linear cross motion components and radial velocities of the stars.
- B. From their radial velocities.
- C. From their reduced proper motions.
- D. From their proper motions and radial velocities in common.

These four cases I shall treat in the order named and discuss the results in section *E* of this chapter.

A. Determination of the velocity ellipsoid from the theoretical linear cross motion components and radial velocities.

13. By the theoretical linear cross motion components of a star I understand those computed by means of equations (4).

Making use of the method of moments, the velocity ellipsoid can be determined as soon as the velocity components of the stars parallel to three axes of a fundamental system of coordinates are known. Shortly before this part of the present investigation was written — it has previously been published in *L.M. I*, 77, but for comparison with other results in this chapter I partly reproduce it here — CHARLIER had defined a galactic system of coordinates based on the results which he had obtained in his examination of the *B*-stars*. This galactic system I then took as a fundamental one, knowing from preceding researches into the motions of the stars that the velocity ellipsoid has its principal axes nearly situated in the galaxy. Though CHARLIER has later on accepted a galactic system of coordinates defined by PICKERING** and used in the following parts of this paper, nothing

* *L.M.* II, 14.

** *H.A.* 56.

In computing the velocities in the G -system I used for all stars that value of R which I obtained from the F -stars brighter than the sixth magnitude, this value being according to table II: $R = 0,69$.

The results obtained in this way I have put together in the following table III.

TABLE III. The solar motion and velocityellipsoid as determined from the theoretical cross motion components and radial velocities.

Magn.	N	m_1 (Sir./St.)	m_2 (Sir./St.)	m_3 (Sir./St.)	Ω_0 (Sir./St.)	L	B	A	D
$m \leq 4,99$	145	$-2,596 \pm 0,266$	$+1,976 \pm 0,403$	$-1,193 \pm 0,244$	$3,474 \pm 0,304$	$198^\circ \pm 5^\circ$	$-25^\circ \pm 5^\circ$	$84^\circ,9 \pm 5^\circ,6$	$-26^\circ,6 \pm 5^\circ,0$
$m \leq 5,99$	203	$-2,811 \pm 0,282$	$+2,236 \pm 0,418$	$-1,493 \pm 0,231$	$3,890 \pm 0,310$	$197^\circ \pm 5^\circ$	$-27^\circ \pm 5^\circ$	$81^\circ,9 \pm 5^\circ,1$	$-26^\circ,5 \pm 4^\circ,6$

TABLE III cont.

α_1					α_2					α_3				
Length (Sir./St.)	l	b	α	δ	Length (Sir./St.)	l	b	α	δ	Length (Sir./St.)	l	b	α	δ
$5,057 \pm 0,297$	$22,9$	0°	$18,2$	$-15^\circ,3$	$3,260 \pm 0,191$	$16,9$	-45°	$2,8$	$-67^\circ,7$	$2,488 \pm 0,146$	$4,9$	-44°	$23,9$	$+17^\circ,8$
$6,166 \pm 0,306$	$22,7$	-2°	$18,3$	$-18^\circ,7$	$3,919 \pm 0,194$	$16,8$	-26°	$6,7$	$-72^\circ,7$	$2,993 \pm 0,148$	$5,0$	-60°	$0,2$	$+2^\circ,0$

Here N is the number of the stars, Ω_0 is the velocity of the sun computed from

$$\Omega_0 = \sqrt{m_1^2 + m_2^2 + m_3^2},$$

where the m -quantities are the stream velocity components or the solar velocity components with reversed signs along the axes of the G -system. L , B and A , D are the coordinates of the antapex, l and b are the coordinates of the points towards which the principal axes α_1 , α_2 , α_3 of the velocity ellipsoid are directed. L , B and l , b are longitude and latitude in PICKERING'S galactic system, whereas A , D are equatorial coordinates.

16. The material used in the preceding paragraph I have also employed for a determination of the velocity distribution along and perpendicular to the projection on the galactic plane of the radius vector from the centre of the Milky Way. The theory for this investigation has also been given by CHARLIER in *L.M. II, 14*, and a reproduction of it is found in *L.M. I, 77*. As to details about it I refer to these memoirs.

In order to use it numerically it is, however, necessary to know the distance of the sun from the centre and the galactic velocity components of the sun. Using for this investigation only F -stars brighter than the fifth magnitude the latter quantities are known from table III. They are as mentioned in paragraph 15 equal to $-m_1$, $-m_2$, $-m_3$ respectively. As to the distance of the sun from the centre of our stellar system it is known from CHARLIER'S investigation of the B -stars*, though the objection may be raised against my using the value of it obtained there

* *L.M. II, 14*.

that it may not be determined with sufficient accuracy, the variations in it as computed from the different *B*-subtypes being rather large.

17. Accepting CHARLIER's value of the position of the centre I got the results in table IV.

TABLE IV. Frequency function of the velocity components in the plane of the galaxy.

Magn.	N	m_{r_1}		m_φ		σ_{r_1}		σ_φ		α_{r_1}		α_φ	
										Length	l	Length	l
$m \leq 4,99$	145	Sir. $+0,044 \pm 0,269$	Sir. $+0,011 \pm 0,401$	Sir. $3,240 \pm 0,190$	Sir. $4,831 \pm 0,284$	Sir. $2,914 \pm 0,171$	Sir. $16,9$	Sir. $5,032 \pm 0,295$	Sir. $22,9$				

Here the indices *r₁* and *φ*, attached to the letters *m*, *σ*, *α*, refer respectively to the directions of the projection of the radius vector on the galactic plane and to that perpendicular to it. The quantities *σ_{r₁}*, *σ_φ* are the dispersions in even these directions, whereas *α_{r₁}*, *α_φ* are the principal axes of that ellipse which represents the distribution of the velocity components here considered. Other notations in table IV are the same as in table III.

B. Determination of the velocity ellipsoid from the radial velocities of the stars.

18. In the preceding section of this chapter we have determined the velocity ellipsoid from the theoretical linear cross motion components *U*, *V*, and radial velocities, *W*, of the stars. By means of certain equations given in *L.M. I*, 77, we were able to calculate the velocity components of the stars referred to three fundamental axes in a direct manner and could from them immediately obtain the moments. In the present case, however, we start from the radial velocities, *W*, alone, and by a least squares solution we find the moments of the velocity components referred to a fundamental system of coordinates. For the same reason as mentioned in paragraph 13 we choose a *galactic* one and now take that defined by PICKERING, which has its

positive *U*- (or *X*-)axis directed towards $\alpha = 280^\circ, \delta = 0^\circ$,
 » *V*- (or *Y*-) » » » $\alpha = 10^\circ, \delta = + 62^\circ$,
 » *W*- (or *Z*-) » » » $\alpha = 190^\circ, \delta = + 28^\circ$.

In the sequel we will call this system *K_g*. The velocity components of the stars referred to it we denote by *U_g*, *V_g*, *W_g*.

19. The problem of determining the velocity ellipsoid from the radial velocities alone was first solved by GYLLENBERG*, but a simplified solution has been

* *L.M. I*, 59; *L.M. II*, 13.

given by CHARLIER in his lectures during the spring term of 1918, and with his kind permission I reproduce it here.

The determination is made in the following three steps:

I. From the second order moments of the data known — i. e. in this case the radial velocities — we compute the second order moments of the velocity components referred to the fundamental system K_g . This is done by the method of least squares.

II. From the moments thus obtained the characteristics of the frequency function referred to the same system are determined.

III. Then the characteristics of the frequency function referred to a system K_1 , whose axes are the principal axes of the velocity ellipsoid, are computed.

Then the problem is solved.

First step. The radial velocity, which we here denote by W_0 , being the only known velocity component of the star, we start from the equation

$$W_0 = \beta_{13} U_g + \beta_{23} V_g + \beta_{33} W_g, \quad (6)$$

where β_{ij} are the direction cosines relating the system K_g to a system, K_0 , having its

positive U_0 -axis in the direction of increasing galactic longitude,

» V_0 - » » » » » » latitude,

» W_0 - » » » » » the line of sight.

W_0 is regarded as positive when the star is moving *from* us. The quantities β_{ij} are the same functions of the galactic longitude, l , and latitude, b , of the stars, as the quantities γ_{ij} used in *L.M. I*, 77 of its α , δ , and are consequently known when l and b of the star are determined. The equation (6) then immediately follows from the scheme

		$K_0:$		
		U_0	V_0	W_0
$K_g:$	U_g	β_{11}	β_{12}	β_{13}
	V_g	β_{21}	β_{22}	β_{23}
	W_g	β_{31}	β_{32}	β_{33}

To the right member of this equation should strictly be added a certain constant, K , this constant being the so-called expansion* detected by CAMPBELL**, but as is known from earlier investigations***, it is extremely small for the F -stars, and I have therefore disregarded it.

For every star, for which the radial velocity has been observed, we now have an equation of the same kind as (6). Squaring it we get — also for each star — the equation

* This term was introduced by CHARLIER.

** Stellar Motions.

*** CAMPBELL: Stellar Motions; GYLLENBERG: L.M. II, 13.

$$W_0^2 = \beta_{13}^2 U_g^2 + \beta_{23}^2 V_g^2 + \beta_{33}^2 W_g^2 + 2\beta_{23}\beta_{33} V_g W_g + 2\beta_{33}\beta_{13} W_g U_g + 2\beta_{13}\beta_{23} U_g V_g,$$

which by a least squares solution gives the numerical values of the means of

$$U_g^2, V_g^2, W_g^2, V_g W_g, W_g U_g, U_g V_g,$$

i. e. the second order moments about the origin of the star's velocity components U_g, V_g, W_g in the fundamental system of coordinates, K_g . These moments we denote by

$$\nu'_{g,11}, \nu'_{g,22}, \nu'_{g,33}, \nu'_{g,23}, \nu'_{g,31}, \nu'_{g,12}.$$

Treating (6) itself by the method of least squares we get the mean values of

$$U_g, V_g, W_g,$$

i. e. the first order moments about the origin of U_g, V_g, W_g . We put

$$\nu'_{g,1} = M(U_g) = m_{g,1},$$

$$\nu'_{g,2} = M(V_g) = m_{g,2},$$

$$\nu'_{g,3} = M(W_g) = m_{g,3}.$$

The required second order moments about the means of U_g, V_g, W_g are then obtained from

$$\nu_{g,ij} = \nu'_{g,ij} - m_{g,i} m_{g,j}, \quad (i, j = 1, 2, 3), \quad (7)$$

and the first step is completed.

The quantities $m_{g,i}$, ($i = 1, 2, 3$), which are the sun's linear velocity components in the system K_g with reversed signs give as a by-product the total linear velocity of the sun and its direction of motion in the usual manner.

Second and third steps. These computations are made by means of the theories given in *L.M. I*, 66 and § 14 of this memoir.

20. In the numerical application of the method sketched in the preceding paragraph I have not followed it literally, but employed the same modifications of it that were used by CHARLIER*, GYLLENBERG** and WICKSELL***, which modifications highly shorten the numerical labour. I have made use of CHARLIER's division of the sky into 48 »squares» of equal area. This division was first made with the equator as fundamental plane but nothing prevents us from using a galactic fundamental plane thus introducing galactic squares. This has also been recently done by CHARLIER, in doing which he uses the galactic system proposed by PICKERING and defined in paragraph 18. The limits of these galactic squares are the same expressed in l and b as those of the equatorial ones expressed in α and δ †,

* *L.M. II*, 9.

** *L.M. II*, 13.

*** *L.M. II*, 12.

† *L.M. II*, 8.

and the direction cosines relating the axes of this galactic system with those of the equatorial system are

		$K_2:$		
		U''	V''	W''
$K_g:$	U_g	ϵ_{11}	ϵ_{21}	ϵ_{31}
	V_g	ϵ_{12}	ϵ_{22}	ϵ_{32}
	W_g	ϵ_{13}	ϵ_{23}	ϵ_{33}

where

$$\begin{aligned} \epsilon_{11} &= +0,1736482; & \epsilon_{12} &= +0,4623393; & \epsilon_{13} &= -0,8695337; \\ \epsilon_{21} &= -0,9848078; & \epsilon_{22} &= +0,0815229; & \epsilon_{23} &= -0,1533223; \\ \epsilon_{31} &= 0; & \epsilon_{32} &= +0,8829476; & \epsilon_{33} &= +0,4694716. \end{aligned}$$

The material available to me being rather scarce I have taken together diametrically situated squares, the radial velocities of the stars in the southern squares then being taken with reversed signs.

One modification is this that the equation (6) is not taken to be valid for a single star, but for a whole square; β_{ij} in the same equation are, as before, the direction cosines relating the system K_g to the systems K_0 , these latter now having their origins in the centres of the squares. (The β_{ij} were tabulated by WICKSELL* and GYLLENBERG** and were called by them γ_{ij} .) For we make the supposition that all stars in a square have the same coordinates as the centre of this square, a supposition that is the more justified the smaller we make the squares. Further W_0 does not mean the radial velocity of a star, but the mean of the radial velocities of the stars in a square (in the sequel denoted by $\bar{W}_0$). Thus we have one equation of the same kind as (6), for every square, consequently in all 48 such equations (or 24, if the squares are taken together in twos).

Proceeding now in the following manner we can do without equation (7). Denoting the mean of a quantity computed from all the stars in a certain square by a bar over it we get from (6) by taking the means of its two membra

$$\bar{W}_0 = \beta_{13} \bar{U}_g + \beta_{23} \bar{V}_g + \beta_{33} \bar{W}_g,$$

and subtracting this equation from (6) we find

$$W_0 - \bar{W}_0 = \beta_{13} (U_g - \bar{U}_g) + \beta_{23} (V_g - \bar{V}_g) + \beta_{33} (W_g - \bar{W}_g). \quad (8)$$

Squaring (8) and then taking the means for the same square of both the membra of the equation thus obtained, we get

$$\begin{aligned} \bar{v}_{0,33} &= \beta_{13}^2 \bar{v}_{g,11} + \beta_{23}^2 \bar{v}_{g,22} + \beta_{33}^2 \bar{v}_{g,33} + 2\beta_{23} \beta_{33} \bar{v}_{g,23} + 2\beta_{33} \beta_{13} \bar{v}_{g,31} + \\ &\quad + 2\beta_{13} \beta_{23} \bar{v}_{g,12}, \end{aligned} \quad (9)$$

where the $\bar{v}$ -quantities denote the moments about the means. Such an equation (9) we get for every square, and making a least squares solution of this set of

* L.M. II, 12.

** L.M. II, 13.

TABLE V. The solar motion and velocity ellipsoid, as determined from the radial velocities.

Magn.	N	$m_{g,1}$ (Sir./St.)	$m_{g,2}$ (Sir./St.)	$m_{g,3}$ (Sir./St.)	Ω (Sir./St.)	L	B	A	D
$m \leq 4,99$	158	$-3,141 \pm 0,474$	$-0,693 \pm 0,479$	$-0,769 \pm 0,568$	$3,255 \pm 0,507$	$191^{\circ},4 \pm 9^{\circ},1$	$-13^{\circ},5 \pm 8^{\circ},8$	$93^{\circ},1 \pm 9^{\circ},2$	$-16^{\circ},3 \pm 8^{\circ},8$
$m \leq 5,99$	243	$-3,288 \pm 0,428$	$-1,768 \pm 0,413$	$-1,087 \pm 0,494$	$3,884 \pm 0,444$	$208,1 \pm 6,8$	$-16,2 \pm 6,6$	$97,7 \pm 7,7$	$-32,1 \pm 6,6$

TABLE V cont.

α_1			α_2			α_3		
Length. (Sir./St.)	l	b	Length. (Sir./St.)	l	b	Length. (Sir./St.)	l	b
$4,808 \pm 0,285$	$23,2 \pm 0,7$	$+13^{\circ},9 \pm 10^{\circ},9$	$2,801 \pm 0,494$	$17,5$	$-18^{\circ},2$	$2,588 \pm 0,701$	$2,8$	$-66^{\circ},6$
$5,014 \pm 0,265$	$21,8 \pm 1,4$	$-2,3 \pm 11,2$	$3,500 \pm 0,361$	$15,8$	$-25,1$	$2,750 \pm 0,600$	$4,2$	$-64,9$
	h	h		h	h		h	h
	$23,2 \pm 0,7$	$17,4$		$17,5$	$10,3$		$2,8$	$23,5$
	$21,8 \pm 1,4$	$17,7$		$15,8$	$6,5$		$4,2$	$23,9$
								$5,3$
								$-11^{\circ},1$

equations we thus get numerical values of the required quantities $\bar{v}_{g,ij}$, ($i, j=1, 2, 3$), valid for the whole sky, and thus denoted by $v_{g,ij}$.

A further modification can be made on account of the circumstance mentioned in paragraph 14. The second step in determining the ellipsoid (paragraph 19) can thus be altered as follows: »From the second order moments, $v_{g,ij}$, of the velocity components referred to the fundamental system K_g , and obtained by the first step, the *moments* of the same order in a system K_1 , whose axes are the principal axes of the ellipsoid, are determined». Then the third step is made simply by taking the inverse square roots of the three moments thus obtained, the other three moments being equal to zero.

21. The number of the F stars varying rather much from one double-square to another, I have in the numerical computation given each equation (9) a weight proportional to the number of stars in the corresponding double square. Treating the stars in two groups according to magnitude I got the results in table V.

The notations are here the same as in the previous tables.

In computing the mean errors in $\alpha_1, \alpha_2, \alpha_3, l$ and b I have made use of the formulæ given by WICKSELL in *L.M. II*, 21 pages 60, 64, 65, 73. The mean errors of L, B were obtained by means of the formulæ given by CHARLIER:*

$$\epsilon(L) = \frac{\epsilon(\Omega_0)}{\Omega_0 \cos B},$$

$$\epsilon(B) = \frac{\epsilon(\Omega_0)}{\Omega_0}.$$

* L.M. II, 9, page 74.

22. The two magnitude groups considered in the preceding paragraph do not include all F stars brighter than the fifth resp. sixth magnitude with known radial velocities. In fact, certain stars, having very large radial velocities, have not been retained in the computation made. As the second order moments of the radial velocities are computed for the stars in each square separately (see paragraph 20) and these moments vary from square to square — just as they should do according to the hypothesis of an ellipsoidal velocity distribution — it seems to me that, in determining which stars are to be rejected there can not be stated an upper limit for the radial velocities of the retained stars valid for the whole sky, which mode was used by GYLLENBERG *, but we must have different limits for the different squares. These limits I have in general put equal to three times the dispersion of the radial velocity in a square, in certain cases I have also excluded stars *within* these limits, but whose rejection considerably altered the dispersion.

C. Determination of the velocity ellipsoid from the reduced proper motions of the stars.

23. Employing also in this section the galactic squares, I denote the galactic proper motion components by

$$u_0 = \Delta l \cos b, \quad v_0 = \Delta b,$$

and the reduced galactic proper motion components by

$$u'_0 = u_0 \cdot 10^{0.2m}, \quad v'_0 = v_0 \cdot 10^{0.2m}.$$

The former have been computed from the equatorial proper motion components, u , v , given in Boss's *Preliminary General Catalogue*. As to the latter they evidently denote the proper motions which a star should have at such a distance that its apparent magnitude is equal to zero.

Making use of modifications analogous to those employed in the section B of this chapter, the theory for computing the velocity ellipsoid from the reduced proper motion components is the following. From equations (4) and the scheme, page 15 of this paper we get

$$\begin{aligned} U_0 &= Ru'_0 = \beta_{11} U_g + \beta_{21} V_g + \beta_{31} W_g, \\ V_0 &= Rv'_0 = \beta_{12} U_g + \beta_{22} V_g + \beta_{32} W_g, \end{aligned} \tag{10}$$

which in this case are our starting equations. Taking the means of both the members of (10) for all the stars in a certain square we find

$$\begin{aligned} \bar{U}_0 &= \beta_{11} \bar{U}_g + \beta_{21} \bar{V}_g + \beta_{31} \bar{W}_g, \\ \bar{V}_0 &= \beta_{12} \bar{U}_g + \beta_{22} \bar{V}_g + \beta_{32} \bar{W}_g, \end{aligned} \tag{11}$$

* L.M. II, 13.

and hence from (10) and (11)

$$\begin{aligned} U_0 - \bar{U}_0 &= \beta_{11}(U_g - \bar{U}_g) + \beta_{21}(V_g - \bar{V}_g) + \beta_{31}(W_g - \bar{W}_g), \\ V_0 - \bar{V}_0 &= \beta_{12}(U_g - \bar{U}_g) + \beta_{22}(V_g - \bar{V}_g) + \beta_{32}(W_g - \bar{W}_g). \end{aligned}$$

Squaring these equations and multiplying them by each other and then taking the means — for the stars in the same square — of both the membra of the three equations thus obtained we get

$$\begin{aligned} \bar{v}_{0,11} &= \beta_{11}^2 \bar{v}_{g,11} + \beta_{21}^2 \bar{v}_{g,22} + \beta_{31}^2 \bar{v}_{g,33} + 2\beta_{21}\beta_{31} \bar{v}_{g,23} + 2\beta_{31}\beta_{11} \bar{v}_{g,31} + 2\beta_{11}\beta_{21} \bar{v}_{g,12}, \\ \bar{v}_{0,22} &= \beta_{12}^2 \bar{v}_{g,11} + \beta_{22}^2 \bar{v}_{g,22} + \beta_{32}^2 \bar{v}_{g,33} + 2\beta_{22}\beta_{32} \bar{v}_{g,23} + 2\beta_{32}\beta_{12} \bar{v}_{g,31} + 2\beta_{12}\beta_{22} \bar{v}_{g,12}, \\ \bar{v}_{0,12} &= \beta_{11}\beta_{12} \bar{v}_{g,11} + \beta_{21}\beta_{22} \bar{v}_{g,22} + \beta_{31}\beta_{32} \bar{v}_{g,33} + (\beta_{21}\beta_{32} + \beta_{22}\beta_{31}) \bar{v}_{g,23} + (\beta_{31}\beta_{12} + \beta_{32}\beta_{11}) \bar{v}_{g,31} + (\beta_{11}\beta_{22} + \beta_{12}\beta_{21}) \bar{v}_{g,12}. \end{aligned} \quad (12)$$

For each square we have three such equations. The unknown quantities required are the moments $\bar{v}_{g,ij}$, ($i, j = 1, 2, 3$), but also the quantities $\bar{v}_{0,ij}$, ($i, j = 1, 2$), in the left membra of (12) are hitherto unknown. However, between the moments $\bar{v}_{0,ij}$, ($i, j = 1, 2$), and the known moments $\bar{\mu}_{0,ij}$, ($i, j = 1, 2$) of the reduced proper motion components there exist certain relations analogous to those given by CHARLIER in *L.M. II*, 9. They are

$$\begin{aligned} \theta_1^2 \bar{v}_{0,11} &= \bar{\mu}_{0,11} q' - \bar{\mu}_{0,1}'^2 (1 - q'), \\ \theta_1^2 \bar{v}_{0,22} &= \bar{\mu}_{0,22} q' - \bar{\mu}_{0,2}'^2 (1 - q'), \\ \theta_1^2 \bar{v}_{0,12} &= \bar{\mu}_{0,12} q' - \bar{\mu}_{0,1}' \bar{\mu}_{0,2}' (1 - q'). \end{aligned} \quad (13)$$

Here is

$$\theta_1 = M \left(\frac{1}{R} \right),$$

$\bar{\mu}_{0,i}'$, ($i = 1, 2$), are the first order moments about the origin (i. e. the means) of the reduced proper motion components, and q' is a certain positive constant always smaller than unity. Inserting now (13) in (12) we get the equations

$$\begin{aligned} \bar{\mu}_{0,11} q' - \bar{\mu}_{0,1}'^2 (1 - q') &= \beta_{11}^2 \theta_1^2 \bar{v}_{g,11} + \beta_{21}^2 \theta_1^2 \bar{v}_{g,22} + \beta_{31}^2 \theta_1^2 \bar{v}_{g,33} + 2\beta_{21}\beta_{31} \theta_1^2 \bar{v}_{g,23} + \\ &\quad + 2\beta_{31}\beta_{11} \theta_1^2 \bar{v}_{g,31} + 2\beta_{11}\beta_{21} \theta_1^2 \bar{v}_{g,12}, \\ \bar{\mu}_{0,22} q' - \bar{\mu}_{0,2}'^2 (1 - q') &= \beta_{12}^2 \theta_1^2 \bar{v}_{g,11} + \beta_{22}^2 \theta_1^2 \bar{v}_{g,22} + \beta_{32}^2 \theta_1^2 \bar{v}_{g,33} + 2\beta_{22}\beta_{32} \theta_1^2 \bar{v}_{g,23} + \\ &\quad + 2\beta_{32}\beta_{12} \theta_1^2 \bar{v}_{g,31} + 2\beta_{12}\beta_{22} \theta_1^2 \bar{v}_{g,12}, \\ \bar{\mu}_{0,12} q' - \bar{\mu}_{0,1}' \bar{\mu}_{0,2}' (1 - q') &= \beta_{11}\beta_{12} \theta_1^2 \bar{v}_{g,11} + \beta_{21}\beta_{22} \theta_1^2 \bar{v}_{g,22} + \beta_{31}\beta_{32} \theta_1^2 \bar{v}_{g,33} + \\ &\quad + (\beta_{21}\beta_{32} + \beta_{22}\beta_{31}) \theta_1^2 \bar{v}_{g,23} + (\beta_{31}\beta_{12} + \beta_{32}\beta_{11}) \theta_1^2 \bar{v}_{g,31} + (\beta_{11}\beta_{22} + \beta_{12}\beta_{21}) \theta_1^2 \bar{v}_{g,12}, \end{aligned} \quad (14)$$

which by a least squares solution give us the quantities $\theta_1^2 \bar{v}_{g,ij}$, ($i, j = 1, 2, 3$), expressed in terms of the unknown q' . Dividing them by the numerical value of θ_1^2 which we have found in chapter I, we get the moments $\bar{v}_{g,ij}$, (or $v_{g,ij}$) ($i, j = 1, 2, 3$), as linear functions of q' .

24. From the examination of the radial velocities we now know the values of $v_{g,ij}$, ($i, j = 1, 2, 3$), and hence, by a comparison between these and the results

obtained in the present section, we can get a numerical value of q' . Indeed we obtain equations of the following kind:

$$\begin{aligned} v_{g,11} &= a_{11} q' + b_{11}; \\ v_{g,22} &= a_{22} q' + b_{22}; \\ v_{g,33} &= a_{33} q' + b_{33}; \\ v_{g,23} &= a_{23} q' + b_{23}; \\ v_{g,31} &= a_{31} q' + b_{31}; \\ v_{g,12} &= a_{12} q' + b_{12}; \end{aligned} \tag{15}$$

where a_{ij} and b_{ij} are known from equations (14) and $v_{g,ij}$ from section *B* of this chapter. From (15) we can solve q' by the method of least squares. However, as we use galactic squares, the correlation moments $v_{g,23}$, $v_{g,31}$, $v_{g,12}$ in general become rather small, and hence we can disregard the last three equations (15).

Another method of getting q' is to add the first three equations (15) and solve the single equation thus obtained. Then in reality we make a comparison between the square of the total mean velocity of the stars obtained from the radial velocities and the same quantity expressed in terms of q' .

When q' is thus known we get the principal axes of the velocity ellipsoid and their directions in the usual manner.

25. Before applying this method to the available material, it proved necessary also in this case to exclude some stars with exceptionally large values of u'_0 or v'_0 . In determining what stars were to be rejected I followed the same rule as in the case of the radial velocities, only choosing $\pm 3,5$ times their dispersions as the upper limits of u'_0 and v'_0 for the retained stars, this for the reason mentioned in paragraph 7. Thus there remained 549 *F*-stars brighter than the sixth magnitude and 165 stars brighter than the fifth magnitude. In order to get more stable values of the second order moments in each square I took together diametrically situated squares, thereby changing the signs of the components u'_0 of the stars in the southern galactic hemisphere. The velocity ellipsoid was computed from the larger group of stars, first by giving each equation (14) the same weight and then assuming the weight proportional to the number of stars in the square. The results were almost identical in these two cases. Table VI contains the results obtained in the former manner. In treating the smaller group of stars I weighted the equations (14) according to the number of stars in the different squares. Making use of the values of R obtained for the two groups of stars here considered and given in table II, I got the results found in table VI. Here I have given the lengths and directions of the principal axes of the velocity ellipsoid for four different values of q' clustering about the values of this quantity found by means of the two methods mentioned in paragraph 24. By the first and second method respectively I got for *F*-stars brighter than the fifth magnitude:

$$\begin{aligned} q' &= 0,672, \\ q' &= 0,698. \end{aligned}$$

TABLE VI. The velocity ellipsoid for different values of q' , as determined from the reduced proper motions.

Magn.	N	q'	α_1					α_2					α_3				
			Length (Sir./St.)	l	b	α	δ	Length (Sir./St.)	l	b	α	δ	Length (Sir./St.)	l	b	α	δ
$m \leq 4.99$	165	0.9	6.445	23.1^h	7.3^0	18.7^h	0^0	14.63, 958	16.6^h	49.8^0	2.7^h	62.8^0	62.8	5.5^h	39.4^0	0.3^h	22.3^0
		0.8	5.939	23.1^h	9.0^0	18.8^h	—	16.52, 860	16.3^h	53.0^0	2.7^h	58.5^0	58.5	5.5^h	36.4^0	0.2^h	25.3^0
		0.7	5.391 $\pm$ 0.167	23.0 ± 0.3	-11.0 ± 4.5^0	18.8^h	—	18.2, 265 $\pm$ 0.429	13.7^h	75.6^0	1.7^h	33.3^0	23.0	5.0^h	6.1^0	23.0^h	53.2^0
		0.6	4.796	22.8^h	-13.7^0	19.0^h	—	21.8, 2, 058	17.2^h	49.3^0	12.6^h	12.7^0	16.8	4.1^h	36.8^0	16.8^h	64.9^0
$m \leq 5.99$	549	0.9	6.633	22.8^h	5.6^0	18.4^h	—	18.3, 4, 997	16.8^h	26.5^0	6.7^h	71.7^0	62.8	5.5^h	62.8^0	0.5^h	0.9^0
		0.8	6.182	22.6^h	7.1^0	18.4^h	—	21.8, 4, 593	16.4^h	26.1^0	6.6^h	68.2^0	62.8	5.5^h	62.8^0	0.5^h	1.0^0
		0.7	5.687	22.3^h	9.0^0	18.4^h	—	26.2, 4, 131	16.0^h	25.3^0	6.6^h	63.7^0	63.0	5.5^h	63.0^0	0.5^h	1.2^0
		0.6	5.165 $\pm$ 0.114	22.6 ± 0.5	-11.2 ± 4.2^0	18.4^h	—	31.7, 3, 579 $\pm$ 0.166	15.7^h	23.8^0	6.6^h	58.2^0	63.2	5.5^h	63.2^0	0.5^h	1.4^0

and 'for F -stars brighter than the sixth magnitude:

$$q' = 0.590,$$

$$q' = 0.585.$$

D. Determination of the velocity ellipsoid from the proper motions and radial velocities of the stars.*

26. The theory of this determination is analogous to those given in sections B and C of the present chapter and consists in making a simultaneous employment of the three equations

$$\begin{aligned} U_0 &= rv_0 = \beta_{11} U_g + \beta_{21} V_g + \beta_{31} W_g, \\ V_0 &= rv_0 = \beta_{12} U_g + \beta_{22} V_g + \beta_{32} W_g, \\ W_0 &= \beta_{13} U_g + \beta_{23} V_g + \beta_{33} W_g, \end{aligned} \quad (16)$$

where the notations are the same as hitherto used. From the stars in a certain square we obtain

$$\begin{aligned} \bar{U}_0 &= \beta_{11} \bar{U}_g + \beta_{21} \bar{V}_g + \beta_{31} \bar{W}_g, \\ \bar{V}_0 &= \beta_{12} \bar{U}_g + \beta_{22} \bar{V}_g + \beta_{32} \bar{W}_g, \\ \bar{W}_0 &= \beta_{13} \bar{U}_g + \beta_{23} \bar{V}_g + \beta_{33} \bar{W}_g \end{aligned} \quad (17)$$

and from (16) and (17) by subtraction

$$\begin{aligned} U_0 - \bar{U}_0 &= \beta_{11}(U_g - \bar{U}_g) + \beta_{21}(V_g - \bar{V}_g) + \beta_{31}(W_g - \bar{W}_g), \\ V_0 - \bar{V}_0 &= \beta_{12}(U_g - \bar{U}_g) + \beta_{22}(V_g - \bar{V}_g) + \beta_{32}(W_g - \bar{W}_g), \\ W_0 - \bar{W}_0 &= \beta_{13}(U_g - \bar{U}_g) + \beta_{23}(V_g - \bar{V}_g) + \beta_{33}(W_g - \bar{W}_g). \end{aligned}$$

Squaring these equations and multiplying them in twos and then taking the means for all stars in the square in question of both the membra of the six equations thus obtained we find

* This examination was undertaken at the request of Dr. GYLLENBERG.

$$\begin{aligned}
 11 &= \beta_{11}^2 \bar{v}_{g,11} + \beta_{21}^2 \bar{v}_{g,22} + \beta_{31}^2 \bar{v}_{g,33} + 2\beta_{21}\beta_{31} \bar{v}_{g,23} + 2\beta_{31}\beta_{11} \bar{v}_{g,31} + 2\beta_{11}\beta_{21} \bar{v}_{g,12}, \\
 22 &= \beta_{12}^2 \bar{v}_{g,11} + \beta_{22}^2 \bar{v}_{g,22} + \beta_{32}^2 \bar{v}_{g,33} + 2\beta_{22}\beta_{32} \bar{v}_{g,23} + 2\beta_{32}\beta_{12} \bar{v}_{g,31} + 2\beta_{12}\beta_{22} \bar{v}_{g,12}, \\
 33 &= \beta_{13}^2 \bar{v}_{g,11} + \beta_{23}^2 \bar{v}_{g,22} + \beta_{33}^2 \bar{v}_{g,33} + 2\beta_{23}\beta_{33} \bar{v}_{g,23} + 2\beta_{33}\beta_{13} \bar{v}_{g,31} + 2\beta_{13}\beta_{23} \bar{v}_{g,12}, \\
 23 &= \beta_{12}\beta_{13} \bar{v}_{g,11} + \beta_{22}\beta_{23} \bar{v}_{g,22} + \beta_{32}\beta_{33} \bar{v}_{g,33} + (\beta_{22}\beta_{33} + \beta_{23}\beta_{32}) \bar{v}_{g,23} + (\beta_{32}\beta_{13} + \beta_{33}\beta_{12}) \bar{v}_{g,31} + (\beta_{12}\beta_{23} + \beta_{13}\beta_{22}) \bar{v}_{g,12}, \\
 31 &= \beta_{13}\beta_{11} \bar{v}_{g,11} + \beta_{23}\beta_{21} \bar{v}_{g,22} + \beta_{33}\beta_{31} \bar{v}_{g,33} + (\beta_{23}\beta_{31} + \beta_{21}\beta_{33}) \bar{v}_{g,23} + (\beta_{33}\beta_{11} + \beta_{31}\beta_{13}) \bar{v}_{g,31} + (\beta_{13}\beta_{21} + \beta_{11}\beta_{23}) \bar{v}_{g,12}, \\
 12 &= \beta_{11}\beta_{12} \bar{v}_{g,11} + \beta_{21}\beta_{22} \bar{v}_{g,22} + \beta_{31}\beta_{32} \bar{v}_{g,33} + (\beta_{21}\beta_{32} + \beta_{22}\beta_{31}) \bar{v}_{g,23} + (\beta_{31}\beta_{12} + \beta_{11}\beta_{32}) \bar{v}_{g,31} + (\beta_{11}\beta_{22} + \beta_{12}\beta_{21}) \bar{v}_{g,12}.
 \end{aligned} \tag{18}$$

In this system of equations the unknown quantities which we want to know are the moments $\bar{v}_{g,ij}$, ($i, j = 1, 2, 3$), but, as in the case of (12), the left membra of (18) except $\bar{v}_{0,33}$ are also unknown to us. However, taking recourse to CHARLIER's relations* between the moments, $\bar{v}_{0,ij}$, of the linear velocity components and the known moments, $\bar{n}_{0,ij}$, of the proper motion components and radial velocities the left membra of (18) become known in terms of two constants ϑ_1 and q' . The necessary relations deduced by CHARLIER are

$$\begin{aligned}
 \vartheta_1^2 \bar{v}_{0,11} &= \bar{n}_{0,11} q' - \bar{n}_{0,1}'^2 (1 - q'), \\
 \vartheta_1^2 \bar{v}_{0,22} &= \bar{n}_{0,22} q' - \bar{n}_{0,2}'^2 (1 - q'), \\
 \vartheta_1^2 \bar{v}_{0,12} &= \bar{n}_{0,12} q' - \bar{n}_{0,1}' \bar{n}_{0,2}' (1 - q'),
 \end{aligned} \tag{19}$$

to which we must add

$$\begin{aligned}
 \vartheta_1^2 \bar{v}_{0,33} &= \vartheta_1^2 \bar{n}_{0,33}, \\
 \vartheta_1^2 \bar{v}_{0,23} &= \vartheta_1^2 \bar{n}_{0,23}, \\
 \vartheta_1^2 \bar{v}_{0,31} &= \vartheta_1^2 \bar{n}_{0,31},
 \end{aligned} \tag{20}$$

which are easily deduced in the same manner as (19). The quantity ϑ_1 is equal to $M\left(\frac{1}{r}\right)$, and q' can be proved to be identical with the q' occurring in section C of this chapter.

From (18), (19), and (20) is now obtained

$$\begin{aligned}
 \bar{n}_{0,11} q' - \bar{n}_{0,1}'^2 (1 - q') &= \beta_{11}^2 \vartheta_1^2 \bar{v}_{g,11} + \beta_{21}^2 \vartheta_1^2 \bar{v}_{g,22} + \beta_{31}^2 \vartheta_1^2 \bar{v}_{g,33} + 2\beta_{21}\beta_{31} \vartheta_1^2 \bar{v}_{g,23} + \\
 &\quad + 2\beta_{31}\beta_{11} \vartheta_1^2 \bar{v}_{g,31} + 2\beta_{11}\beta_{21} \vartheta_1^2 \bar{v}_{g,12}, \\
 \bar{n}_{0,22} q' - \bar{n}_{0,2}'^2 (1 - q') &= \beta_{12}^2 \vartheta_1^2 \bar{v}_{g,11} + \beta_{22}^2 \vartheta_1^2 \bar{v}_{g,22} + \beta_{32}^2 \vartheta_1^2 \bar{v}_{g,33} + 2\beta_{22}\beta_{32} \vartheta_1^2 \bar{v}_{g,23} + \\
 &\quad + 2\beta_{32}\beta_{12} \vartheta_1^2 \bar{v}_{g,31} + 2\beta_{12}\beta_{22} \vartheta_1^2 \bar{v}_{g,12}, \\
 \vartheta_1^2 \bar{n}_{0,33} &= \beta_{13}^2 \vartheta_1^2 \bar{v}_{g,11} + \beta_{23}^2 \vartheta_1^2 \bar{v}_{g,22} + \beta_{33}^2 \vartheta_1^2 \bar{v}_{g,33} + 2\beta_{23}\beta_{33} \vartheta_1^2 \bar{v}_{g,23} + \\
 &\quad + 2\beta_{33}\beta_{13} \vartheta_1^2 \bar{v}_{g,31} + 2\beta_{13}\beta_{23} \vartheta_1^2 \bar{v}_{g,12}, \\
 \vartheta_1^2 \bar{n}_{0,23} &= \beta_{12}\beta_{13} \vartheta_1^2 \bar{v}_{g,11} + \beta_{22}\beta_{23} \vartheta_1^2 \bar{v}_{g,22} + \beta_{32}\beta_{33} \vartheta_1^2 \bar{v}_{g,33} + (\beta_{22}\beta_{33} + \beta_{23}\beta_{32}) \vartheta_1^2 \bar{v}_{g,23} + \\
 &\quad + (\beta_{32}\beta_{13} + \beta_{33}\beta_{12}) \vartheta_1^2 \bar{v}_{g,31} + (\beta_{12}\beta_{23} + \beta_{13}\beta_{22}) \vartheta_1^2 \bar{v}_{g,12}, \\
 \vartheta_1^2 \bar{n}_{0,31} &= \beta_{13}\beta_{11} \vartheta_1^2 \bar{v}_{g,11} + \beta_{23}\beta_{21} \vartheta_1^2 \bar{v}_{g,22} + \beta_{33}\beta_{31} \vartheta_1^2 \bar{v}_{g,33} + (\beta_{23}\beta_{31} + \beta_{21}\beta_{33}) \vartheta_1^2 \bar{v}_{g,23} + \\
 &\quad + (\beta_{33}\beta_{11} + \beta_{31}\beta_{13}) \vartheta_1^2 \bar{v}_{g,31} + (\beta_{13}\beta_{21} + \beta_{11}\beta_{23}) \vartheta_1^2 \bar{v}_{g,12}, \\
 \bar{n}_{0,12} q' - \bar{n}_{0,1}' \bar{n}_{0,2}' (1 - q') &= \beta_{11}\beta_{12} \vartheta_1^2 \bar{v}_{g,11} + \beta_{21}\beta_{22} \vartheta_1^2 \bar{v}_{g,22} + \beta_{31}\beta_{32} \vartheta_1^2 \bar{v}_{g,33} + (\beta_{21}\beta_{32} + \beta_{22}\beta_{31}) \vartheta_1^2 \bar{v}_{g,23} + \\
 &\quad + (\beta_{31}\beta_{12} + \beta_{11}\beta_{32}) \vartheta_1^2 \bar{v}_{g,31} + (\beta_{11}\beta_{22} + \beta_{12}\beta_{21}) \vartheta_1^2 \bar{v}_{g,12}.
 \end{aligned} \tag{21}$$

* L.M. II, 9.

Having for each square a system (21) we get by the method of least squares the quantities $\vartheta_1^2 \bar{v}_{g,ij}$ expressed in terms of q' , provided we know the value of ϑ_1 , which quantity is still present in the left membrum of the third, fourth, and fifth equations of (21).

It is, however, obtained in the following manner. The first two equations (16) give

$$\begin{aligned} u_0 &= \beta_{11} \cdot \frac{U_g}{r} + \beta_{21} \cdot \frac{V_g}{r} + \beta_{31} \cdot \frac{W_g}{r}, \\ v_0 &= \beta_{12} \cdot \frac{U_g}{r} + \beta_{22} \cdot \frac{V_g}{r} + \beta_{32} \cdot \frac{W_g}{r}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \bar{n}'_{0,1} &= \beta_{11} \vartheta_1 \bar{v}'_{g,1} + \beta_{21} \vartheta_1 \bar{v}'_{g,2} + \beta_{31} \vartheta_1 \bar{v}'_{g,3}, \\ \bar{n}'_{0,2} &= \beta_{12} \vartheta_1 \bar{v}'_{g,1} + \beta_{22} \vartheta_1 \bar{v}'_{g,2} + \beta_{32} \vartheta_1 \bar{v}'_{g,3}, \end{aligned} \quad (22)$$

provided the velocities are independent of the distances. Such a system (22) we have for each square, and thus by a least squares solution we obtain numerical values of the quantities $\vartheta_1 \bar{v}'_{g,i}$, ($i = 1, 2, 3$), which are the galactic angular velocity components of the sun with reversed signs. Squaring and adding them up and taking the square root of the sum, we find the value of the total angular velocity, $\vartheta_1 \Omega_0$, of the sun. But from the third equation (16) we get the solar velocity, Ω_0 , in absolute measure according to section *B* of this chapter, and thus ϑ_1 can be computed. Inserting this value in the left membra of (21) before applying the method of least squares to this system, the quantities $\vartheta_1^2 \bar{v}_{g,ij}$ are then obtained in terms of q' alone

The mode of our further proceeding is then quite analogous to that we used after solving the system (15) in paragraph 24.

27. It was mentioned in the preceding paragraph that the constant q' has the same definition both in the case of reduced proper motions and in case of common proper motions. This can be shown in the following manner. For the mean value of the inverse of the s^{th} power of r for stars having their magnitudes between the limits $m + \frac{1}{2} dm$ and $m - \frac{1}{2} dm$ CHARLIER* has given the expression

$$\vartheta_s = M_m \left(\frac{1}{r^s} \right) = \int_{-\infty}^{+\infty} dy e^{sby} \Delta_0(y) \varphi_0(m+y) : a(m) \quad (23)$$

where

$$a(m) = \int_{-\infty}^{+\infty} dy \Delta_0(y) \varphi_0(m+y), \quad (24)$$

* L.M. II, 9.

and denotes the number of stars having their magnitudes between the limits above,

$$b = \frac{0,2}{10 \log e} = 0,4605,$$

$$y = -5 \cdot 10 \log r.$$

Assuming Δ_0 and φ_0 to have the forms

$$\Delta_0(y) = \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(y-m_1)^2}{2\sigma_1^2}},$$

$$\varphi_0(m+y) = \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(m+y-m_2)^2}{2\sigma_2^2}},$$

CHARLIER obtains

$$\vartheta_s = p^s q^{s^2} = p^s q'^{-2s^2},$$

or

$$\log \vartheta_s = s \log p - s^2 \log q'^2 \quad (25)$$

where

$$q = \frac{1}{q'^2} = e^{\frac{1}{2} b^2 k^2 \lambda_1 (1 - \lambda_1)}, \quad k^2 = \sigma_1^2 + \sigma_2^2, \quad \lambda_1 = \frac{\sigma_1^2}{k^2}.$$

We have now put

$$\theta_s = M_m \left(\frac{1}{R^s} \right)$$

and from (3) we get

$$\frac{1}{R^s} = \frac{10^{0,2sm}}{r^s} = \frac{e^{sbm}}{r^s}.$$

Thus by means of (24) we find

$$\theta_s = M_m \left(\frac{1}{R^s} \right) = \int_{-\infty}^{+\infty} dy e^{sby} e^{sbm} \Delta_0(y) \varphi_0(m+y) : a(m). \quad (26)$$

From this it is seen that the factor e^{sbm} , which constitutes the only difference between the equations (26) and (23), only alters the coefficient of s in the equation (25), whereas the coefficient of s^2 , which according to (25) is a function of q' , remains unaltered. Hence q' itself is also unaltered.

28. As radial velocity determinations are rather few among the stars of the fifth magnitude, but on the other side nearly all stars brighter than this magnitude have observed radial velocities, I confined the numerical application of the preceding theory to these latter stars. Their number being only 158, I took together diametrically situated squares and assigned to the equations (21) and (22) weights proportional to the number of the stars in the corresponding double square. Just as in sections *B* and *C* of this chapter, certain stars have here been excluded on

TABLE VII. The velocity ellipsoid for different values of q' , as determined from the proper motions and radial velocities.

Magn.	N	q'	a_1					a_2					a_3				
			Length (Sir./St.)	l	b	α	δ	Length (Sir./St.)	l	b	α	δ	Length (Sir./St.)	l	b	α	δ
$m \leq 4.90$	158	0.9	5,138	$^{h}_{22.8}$	$+16^{\circ}.4$	$^{h}_{17.1}$	$-7^{\circ}.7$	3,620	$^{h}_{16.3}$	$+24^{\circ}.1$	$^{h}_{10.8}$	$-32^{\circ}.1$	3,165	$^{h}_{18.8}$	$-59^{\circ}.7$	$^{h}_{0.0}$	$-57^{\circ}.4$
		0.8	4,862	$^{h}_{22.7}$	$+15^{\circ}.7$	$^{h}_{17.1}$	$-9^{\circ}.0$	3,432	$^{h}_{16.2}$	$+25^{\circ}.4$	$^{h}_{10.8}$	$-30^{\circ}.7$	2,956	$^{h}_{18.7}$	$-59^{\circ}.9$	$^{h}_{0.0}$	$-57^{\circ}.3$
		0.7	4,569	$^{h}_{22.6 \pm 0.5}$	$+14.7 \pm 6^{\circ}.6$	$^{h}_{17.1}$	$-10^{\circ}.6$	$3,227 \pm 0.222$	$^{h}_{16.1}$	$+26^{\circ}.7$	$^{h}_{10.8}$	$-29^{\circ}.0$	$2,726 \pm 0.257$	$^{h}_{18.7}$	$-60^{\circ}.6$	$^{h}_{0.1}$	$-56^{\circ}.8$
		0.6	4,260	$^{h}_{22.5}$	$+13^{\circ}.6$	$^{h}_{17.1}$	$-12^{\circ}.5$	3,002	$^{h}_{16.0}$	$+27^{\circ}.9$	$^{h}_{10.7}$	$-27^{\circ}.0$	2,433	—	—	—	—

account of very large values of u_0 , v_0 or W_0 . They were picked out according to the rule mentioned in paragraph 22. The upper limits for the values of u_0 and v_0 of the retained stars were, however, fixed at ± 3.5 times the dispersions of these variates on account of the great excess that is found in their frequency curves.

Neglecting the expansion K and the motion of the invariable plane, which latter is, as I found, very badly determined, the number of stars being rather small, I got from the first order moments the total angular velocity of the sun

$$\vartheta_1 \Omega_0 = 0''.1624 \pm 0''.0335 \text{ per year}$$

and the coordinates of the antapex

$$L = 196^{\circ}.2 \pm 14^{\circ}.8,$$

$$B = -36^{\circ}.9 \pm 11^{\circ}.8,$$

or

$$A = 71^{\circ}.0 \pm 13^{\circ}.5,$$

$$D = -28^{\circ}.6 \pm 11^{\circ}.8.$$

In section *B* of this chapter I found from the same stars that are used here the absolute velocity of the sun

$$\Omega_0 = 3,295 \pm 0,507 \text{ Sir./St.}$$

Thus it is obtained

$$\vartheta_1 = 0''.0493 \pm 0''.0264.$$

The mean error of ϑ_1 (denoted by $\varepsilon(\vartheta_1)$) I computed by means of the formula

$$\varepsilon^2(\vartheta_1) = \varepsilon^2(\vartheta_1 \Omega_0) \cdot \frac{1}{\Omega_0^2} + \varepsilon^2(\Omega_0) \cdot \frac{(\vartheta_1 \Omega_0)^2}{\Omega_0^4},$$

which is easily deduced from the equation

$$\vartheta_1 = \frac{\vartheta_1 \Omega_0}{\Omega_0}.$$

With this value of ϑ_1 I found the lengths of the principal axes and the axial directions for different values of q' as given in table VII.

The value of q' itself was found, by means of the two methods of determining it given in paragraph 24, to be

$$q' = 0,725,$$

$$q' = 0,697.$$

E. Discussion of the results obtained in this chapter.

29. In the following discussion I shall, for the sake of simplicity, refer to the stars brighter than the fifth magnitude as group I, and to the stars brighter than the sixth magnitude as group II. Further I shall call the four kinds of material used in sections *A*, *B*, *C*, *D* of this chapter the materials *A*, *B*, *C*, *D* respectively, and the ellipsoids obtained from these materials will be called the ellipsoids *A*, *B*, *C*, *D* respectively.

Looking first at the results obtained from the first order moments of the velocities of the stars, i. e. Ω_0 , $\vartheta_1\Omega_0$, ϑ_1 , *A*, and *D*, we immediately see that group I yields a smaller value of the solar velocity, Ω_0 , than group II. This fact results both from material *A* and material *B*. The cause of this phenomenon is probably due to the following circumstance. Group II consists of two parts, viz. group I and group III, this latter including stars of the fifth magnitude. Nearly all stars in group I have known radial velocities, whereas this is the case only with about 14 per cent of the stars in group III. Thus the radial velocity stars of this latter group are suspected not to be well suited for employing in a determination of the solar motion and velocity ellipsoid, because they cannot be representative of the whole mass of the stars of this group, especially as radial velocity stars of, and fainter than, the fifth magnitude are not chosen at random, but are chiefly stars with large proper motions, this property making probable a great radial velocity, which produces a large, and consequently easily measured, alteration of the spectral lines.

The rather low value of Ω_0 obtained from group I of materials *A* and *B* — this value being about 3,4 Sir./St. — agrees with that obtained by CAMPBELL * from 185 *F*-stars:

$$\Omega_0 = 15,8 \text{ km./sec.} = 3,3 \text{ Sir./St.}$$

and nearly with that deduced by L. Boss **:

$$\Omega_0 = 17,7 \text{ km./sec.} = 3,7 \text{ Sir./St.}$$

Accepting the hypothesis that the peculiar motions of the stars are mainly in the galactic plane (see paragraph 11) PLUMMER *** finds from 127 *F*-stars of all magnitudes

$$\Omega_0 = 18,3 \text{ km./sec.} = 3,9 \text{ Sir./St.}$$

Treating 237 *F*-stars of all available magnitudes GYLLENBERG † got

$$\Omega_0 = 4,11 \text{ Sir./St.}$$

* CAMPBELL: Stellar Motions, page 209.

** Astronomical Journal, vol. 28. (briefly A. J. 28.)

*** M.N. 76.

† L.M. II, 13.

30. The value of the angular solar velocity I have found from group I, material *D*, to be

$$\vartheta_1 \Omega_0 = 0'',1624 \pm 0'',0335 \text{ per year.}$$

This is a rather large value of this quantity, and its excess over that found by GYLLENBERG *:

$$\vartheta_1 \Omega_0 = 0'',1230 \text{ per year}$$

may be due to the fact that in determining it I have employed only radial velocity stars brighter than the fifth magnitude, and the number of these stars is not the same as the number of all stars of this magnitude-group. Further the selection of stars for radial velocity determinations may also be a cause conducive to the disagreement in question.

31. Probably for the same reason I have got a rather high value of ϑ_1 , viz.

$$\vartheta_1 = 0'',0493.$$

However, the main reason why this quantity has turned out so large is that in computing it from $\vartheta_1 \Omega_0$ I have used the value of $\Omega_0 = 3,295$ Sir./St. — found in section *B* of this chapter from the same stars as those which have yielded the value of $\vartheta_1 \Omega_0$ itself. Using the solar velocity

$$\Omega_0 = 4,110 \text{ Sir./St.}$$

found by GYLLENBERG *, I got

$$\vartheta_1 = 0'',0395,$$

which, as will be seen in the sequel is still larger than all values of this quantity found by other investigators. Indeed, employing the above value of the solar velocity found by himself GYLLENBERG got from 128 radial velocity stars of type *F* brighter than the fifth magnitude

$$\vartheta_1 = 0'',0270$$

and from another solution

$$\vartheta_1 = 0'',0299,$$

whereas CAMPBELL ** from 180 *F*-stars of all magnitudes found

$$\vartheta_1 = 0'',0354,$$

and CHARLIER and GYLLENBERG *** from 142 *F*-stars brighter than the fifth magnitude

$$\vartheta_1 = 0'',0325.$$

32. Concerning the coordinates of the apex it is seen from the tables that all the four materials yield the well known values of them. From the material *B*,

* L.M. II, 13.

** Lick Bulletin, n:o 196.

*** L.M. I, 91.

however, results a large difference between the declination of the apex as obtained from group I and group II, the latter group giving a considerably higher value of this quantity than the former group. As group I in the material in question contains practically all stars brighter than the fifth magnitude we may regard the value $D = +16^{\circ},3$ given by it as safe. The large difference mentioned above may then be explained in the following manner, this explanation being given by CHARLIER. The right ascension of the principal vertex is, as is well known, approximately the same as that of the apex, whereas the declinations of these points are also approximately the same but have opposite signs, the declination of the apex being positive. Assuming now that group II of material *B* contains a greater proportion of stars moving towards the vertex than group I of the same material, this must tend to lower the declination of the apex as given by group I. This may also be explained from the reason mentioned in paragraph 29 that group II of material *B* is not representative of group II as a whole.

PLUMMER* finds for the coordinates of the apex:

$$A = 265^{\circ},0, \quad D = +13^{\circ},1,$$

thus a still lower value of the declination than found by me, and L. Boss** gets

$$A = 263^{\circ},6, \quad D = +15^{\circ},7.$$

Further CHARLIER and GYLLENBERG*** obtain from 508 proper motion stars brighter than the sixth magnitude

$$A = 269^{\circ},0, \quad D = +24^{\circ},6.$$

33. We now proceed to the results derived from the second order moments, and first look at the lengths of the principal axes of the velocity ellipsoids. From the tables III, V, VI, and VII it is immediately seen 1:o: that one axis is considerably longer than the two others and 2:o: that these latter are of nearly equal lengths especially when their mean errors are taken into consideration. These both facts are of very great interest. The first of them shows that the *F*-stars have a rather strongly marked preferential motion and this motion is according to the same tables as referred to above approximately parallel to the galactic plane. The point towards which it is directed is the principal vertex. Together with the second fact it supports the monistic conception† of the stellar universe. CHARLIER†† has shown

- 1) that, if we consider our stellar system as having a shape symmetrical about a certain axis and

* M.N. 76.

** A.J. 28.

*** L.M. I, 91.

† L.M. I, 81.

†† L.M. I, 82

- 2) if we assume a star in this system as influenced only by the total attraction of the whole system,
- 3) if further in this system dynamical equilibrium prevails and
- 4) the velocity surface is an ellipsoid, then
 - I) this ellipsoid must be a rotational one having
 - II) its longest axis as the axis of revolution and
 - III) directed perpendicular to the radius vector from the centre of the stellar system.

The supposition 1) is shown by CHARLIER* to be true for the *B*-stars, and we can, with a certain degree of probability, assume the same about stars of other types. The assumption 4) has been shown by several investigators of the velocities of the stars and in the present paper to agree with reality, but 2) and 3) are mere abstractions. That the conclusions I) and II) are *nearly* true is shown in the foregoing paragraphs. This is also the case with the conclusion III), as we shall see at once. In his investigation of the *B*-stars CHARLIER* has determined the centre of our stellar system. Computing the angle ψ between the radius vector from the sun to this centre and the greatest axis of the velocity ellipsoid I get the following values:

from group I of material *A*: $\psi = 70^{\circ},6$,

from the results of table VI: $\psi = 69^{\circ},8$.

Other ellipsoids in our tables, of course, give approximately the same value of ψ , as the direction of the longest axis is markedly constant for all the ellipsoids computed. Thus the conclusion III) also seems to be *nearly* true.

Hence we can conclude:

a) that the force acting on a star of our stellar system is *chiefly* produced by the total attraction of the whole system, the attractions of the particular stars playing only a subordinate rôle, and

b) that the mass of the *F*-stars is *nearly* in dynamical equilibrium.

34. The tables III, IV, and V show that α_r — i. e. the dispersion of the velocity components along the projection of the radius vector from the centre of the galaxy on the galactic plane — is approximately equal to α_3 . This is a confirmation of CHARLIER's** supposition that from an investigation of stars of later spectral types this very property would appear.

35. Another fact shown by the tables III, V, VI, and VII is that the least axis of the velocity ellipsoid is in most cases directed approximately towards the galactic pole. The exceptions are the ellipsoids computed from group I of materials *A* and *C*. The same result has also been obtained by EDDINGTON and HARTLEY*** as well as by CHARLIER and GYLLENBERG†, whereas GYLLENBERG†† from his examination of the radial velocities of the *F*- and *G*-stars in common as well as of

* L.M. II, 14.

** L.M. II, 14, page 101.

*** M.N. 75.

† L.M. I, 91.

†† L.M. II, 13.

the *F*-stars alone finds ellipsoids whose *median* axes point towards the pole of the galaxy. Now EDDINGTON'S and HARTLEY'S method for computing the velocity ellipsoid seems somewhat obscure to me. They start from the equation

$$|W_0| = \gamma_{13}^2 U_g^2 + \gamma_{23}^2 V_g^2 + \gamma_{33}^2 W_g^2 + 2\gamma_{23}\gamma_{33} V_g W_g + 2\gamma_{33}\gamma_{13} W_g U_g + 2\gamma_{13}\gamma_{23} U_g V_g,$$

where $|W_0|$ is the radial velocity regardless of sign, and GYLLENBERG uses the strictly right equation

$$W_0^2 = \gamma_{13}^2 U_g^2 + \gamma_{23}^2 V_g^2 + \gamma_{33}^2 W_g^2 + 2\gamma_{23}\gamma_{33} V_g W_g + 2\gamma_{33}\gamma_{13} W_g U_g + 2\gamma_{13}\gamma_{23} U_g V_g.$$

To this circumstance *may* be due the disagreement between the results of these investigators, but by means of GYLLENBERG'S method I have from the radial velocities got the same result as EDDINGTON and HARTLEY regarding the shape of the ellipsoid. The discrepancy between GYLLENBERG'S result and mine in this case I have not been able to explain, but perhaps it is a consequence of the different modes used by us in excluding stars with large radial velocities.

36. The excentricity of the velocity ellipsoid is somewhat variable from one material to the other, but within the same material it is approximately the same regardless of the magnitude group. I got the following values of the proportions between the axes:

from material A, group I:	2,03 : 1,31 : 1,00,
» » » , » II:	2,06 : 1,31 : 1,00,
» » B, » I:	1,86 : 1,08 : 1,00,
» » » , » II:	1,82 : 1,27 : 1,00,
» » C, » I, $q' = 0,7$:	2,56 : 1,08 : 1,00,
» » » , » II, $q' = 0,6$:	2,03 : 1,40 : 1,00,
» » D, » I, $q' = 0,7$:	1,68 : 1,18 : 1,00,

CHARLIER and GYLLENBERG got for $q' = 0,75$: 1,81 : 1,29 : 1,00,

EDDINGTON and HARTLEY got: 2,25 : 1,34 : 1,00.

TABLE VIII. Values of e_{12} and e_{13} .

Material	Magn.	q'	e_{12}	e_{13}		
C	$m \leq 4,99$	0,9	0,85	0,93		
		0,8	0,88	0,93		
		0,7	0,90	0,92		
		0,6	0,90	0,96		
		0,9	0,66	0,88		
	$m \leq 5,99$	0,8	0,67	0,88		
		0,7	0,69	0,87		
		0,6	0,72	0,87		
		D	$m \leq 4,99$	0,9	0,71	0,79
				0,8	0,71	0,79
0,7	0,71			0,80		
0,6	0,71			0,81		

From the material C and D have been obtained the values contained in table VIII of the excentricities e_{12} , e_{13} , in the $\alpha_1\alpha_2$ - and $\alpha_1\alpha_3$ -planes of the ellipsoids, which confirm the fact first pointed out by WICKSELL* that the shape of the velocity ellipsoid is almost independent of the value of q' .

37. As to the total mean velocity of the stars, $\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, it is immediately seen from tables III and V that this quantity is obtained greater from group II than from group I.

* L.M. II, 12.

Indeed we get

from material *A*, group I, the total mean velocity equal to 6,511 Sir./St.

»	»	<i>A</i> ,	»	II,	»	»	»	»	»	»	»	7,895	»
»	»	<i>B</i> ,	»	I,	»	»	»	»	»	»	»	6,131	»
»	»	<i>B</i> ,	»	II,	»	»	»	»	»	»	»	6,705	»

This fact may be due to the mode of selection of stars of group II for radial velocity determinations.

For the total mean velocity has previously been found

by GYLLENBERG *	6,865 Sir./St. (from the <i>F</i> - and <i>G</i> -stars),
	6,557 » (from the <i>F</i> -stars),
» EDDINGTON and HARTLEY **	5,478 » (» » »).

These values agree with those found by me except that given by EDDINGTON and HARTLEY and the disagreement may be a consequence of the different methods used by them, and by CHARLIER, GYLLENBERG, and myself in computing the velocity ellipsoid.

38. Considering now the directions of the principal axes of the velocity ellipsoids, it will be seen at the first glance at the tables III, V, VI, and VII that the ellipsoids are, in general, not well situated in the galaxy. Especially is this the case with the ellipsoids *A* and *C* deduced from group I. The reason of this may in the case of the former ellipsoid partly be found in the value of *R* used in computing the linear cross motion components *U* and *V*. (See section *A* of this chapter.)

However, even the axes of the other ellipsoids computed by me do not very well define a galactic system, and this result has also been obtained by other investigators on the motions of the *F*-stars. For comparison I give below the results in this respect obtained by CHARLIER and GYLLENBERG *** from the proper motions of 508 *F*-stars brighter than the sixth magnitude, and by EDDINGTON and HARTLEY ** from 199 *F*-stars with known radial velocities. These investigators found:

	α_1		α_2		α_3		
	l	b	l	b	l	b	
CHARLIER and GYLLENBERG	22,5 ^h	— 10,9 ^o	15,5 ^h	— 53,5 ^o	5,0 ^h	— 34,3 ^o	(for $q' = 0,75$)
EDDINGTON and HARTLEY	22,7	0	20,7	— 37	4,2	— 59	

(The coordinates of the axial directions being given in α and δ by EDDINGTON and HARTLEY they have been plotted on a globe and the galactic coordinates measured.) These ellipsoids too are badly orientated in respect to the galaxy, and their deviations from it are approximately the same as those obtained by me.

* L.M. II, 13.

** M.N. 75.

*** L.M. I, 91.

The coordinates of the principal vertex are in most cases found by me to agree with previous results in this respect. For the *F*-stars they have before been determined only by EDDINGTON and HARTLEY *, and by CHARLIER and GYLLENBERG **, and the values obtained by them I have given on page 32.

A remarkable fact is however that from material *D* the galactic latitude of the principal vertex is found to be positive, whereas the materials *A*, *B*, and *C* yield an approximately equal but negative value of this coordinate. This difference I at first was inclined to ascribe to my using the large value of ϑ_1 obtained in paragraph 28, but recomputing the position of the vertex for $q' = 0,7$ by employing $\vartheta_1 = 0'',0395$ — which value is got by means of the solar velocity $\Omega_0 = 4,11$ Sir./St. *** — its latitude turned out even a little greater than when I used $\vartheta_1 = 0'',0493$. It thus seems to me that the employment of the third, fourth, and fifth equations of (21) is the cause of this deviation from other results.

39. Concerning the constant q' it has previously been determined for the *F*-stars only by CHARLIER and GYLLENBERG **. They computed it by means of the second method mentioned in paragraph 24 and obtained the value

$$q' = 0,690.$$

By the first and second method in the same paragraph I got the values of it contained in the small table below.

Hence we can regard q' to have a value of about 0,6 or 0,7 for the *F*-stars. The values of this quantity computed in these manners are, however, very uncertain. In fact, both membra of equations (15) being obtained by least squares solutions, they have certain mean errors, which, according to the relatively small number of stars available to me, are rather great. As now q' itself is obtained from (15) by a least squares solution it is obvious that its mean error must

TABLE IX. Values of q' .

Material	Magn.	q'	
		Method I	Method II
<i>C</i>	$m \leq 4,99$	0,672	0,698
<i>C</i>	$m \leq 5,99$	0,590	0,586
<i>D</i>	$m \leq 4,99$	0,725	0,697
<i>C, D</i>	$m \leq 4,99$	0,681	0,698
<i>C</i> {	$m \leq 5,99$ {	0,615	0,640
<i>D</i> }	$m \leq 5,99$ }		

be large. Under such circumstances one must be content, if the value of q' is not obtained greater than unity — which it cannot theoretically exceed † — and if it does not fall short of say + 0,5.

The value of q' for stars of all spectral types was determined by WICKSELL †† by a quite different method from that used here. He found that it ought to lie somewhere about 0,75.

* M.N. 75.

** L.M. I, 91.

*** Compare paragraph 31.

† See L.M. II, 9.

†† L.M. II, 12.

CHAPTER III.

Examination of the parallax stars.

40. This chapter is wholly devoted to an investigation of those *F*-stars for which the parallaxes are known. Such stars are chiefly found in three catalogues, namely:

I. *Astrophysical Journal* vol. 46, which contains five hundred stars of different spectral types, for which ADAMS and JOY have determined the parallaxes by means of ADAMS' spectroscopical method.

II. WALKEY's catalogue of parallax stars found in the appendix to the *Journal of the British Astronomical Association*, vol. XXVII. This is only a compilation from other scattered sources of parallaxes determined in the usual manner.

III. *Publications of the Astronomical Laboratory at Groningen* No 24, in which KAPTEYN and WEERSMA give 360 stars with known parallaxes.

These catalogues are in the sequel referred to as catalogues I, II, and III respectively.

41. In catalogue I, however, are given the spectral types for every star determined in three manners. First those given in *H. A. 50*, then the »estimated» and at last the »measured» spectral types. (Concerning the terminology I refer to the catalogue itself.) Transforming the spectral types into spectral indices as defined by CHARLIER in *L.M. II*, 19, I decided to use the spectral types corresponding to the spectral indices

$$\frac{\text{»Estimated» spectral index} + \text{»Measured» spectral index}}{2}$$

and this on the following grounds. The »estimated» spectral types and those given in *H. A. 50* are mostly concordant, as they must be, because the methods used in determining them are principally the same. However, in order to determine the intensities of the spectral lines — which are the basis for the computation of the parallaxes with ADAMS' method — the spectra must have been more carefully examined by ADAMS than they probably were at Harvard, and thus the »estimated» spectral types may be regarded as more reliable than those given in *H. A. 50*. Now, in order to make use also of the method of ADAMS for determining the spectral types — which is undoubtedly of great value — I have accepted the spectral types determined by

means of the above formula. As catalogues II and III contain spectra given in *H. A. 50* and parallaxes measured »trigonometrically», the whole material used by me appears to be not exactly homogeneous. I have therefore treated it in two groups, the first (group α) containing stars found in catalogue I and the second (group β) those contained in catalogue II, but not in catalogue I together with those found in the third, but not in the first two catalogues. Lastly I have examined all the parallax stars as a whole.

42. The purpose of the investigation of the parallax stars being to get an idea of to what extent the absolute magnitude, M , of the F -stars are related to some other properties of the same stars, I employed the multiple correlation method as found in the monograph of CHARLIER, *L.M. I, 66*.

The properties — except absolute magnitude M — to which I paid attention were the apparent magnitude, m , and total cross motion, μ . As, however, this latter property has a frequency curve of type B^* , but the frequency curves of M and m are supposed to be normal (i. e. of type A^*), I also sought the correlation between M and $\log \mu (= \mu_0)$, and between M on one side and m and μ_0 on the other.

From the values of the parallax I first computed the distance in *Sr.* by means of formula (5) and then the absolute magnitude from

$$M = m - 5 \log r.$$

Putting the values of M and m , M and μ , M and μ_0 , into correlation tables, which space does not allow of reproducing here, and doing this separately with the stars from the groups α , β , and both, I got the results found in the tables below.

TABLE X. Correlation between M , m , and μ .

Group.	N	m_1	m_2	m_3	σ_1	σ_2	σ_3
		m m	m m	m m	m m	m m	m m
α	134	$+2,145 \pm 0,124$	$+5,590 \pm 0,136$	$+0,405 \pm 0,026$	$1,440 \pm 0,088$	$1,576 \pm 0,096$	$0,298 \pm 0,018$
β	33	$+2,776 \pm 0,307$	$+5,939 \pm 0,263$	$+0,711 \pm 0,089$	$1,761 \pm 0,217$	$1,512 \pm 0,186$	$0,513 \pm 0,063$
$\alpha + \beta$	165	$+2,262 \pm 0,120$	$+5,684 \pm 0,122$	$+0,445 \pm 0,025$	$1,536 \pm 0,085$	$1,561 \pm 0,086$	$0,324 \pm 0,018$

TABLE X. Contin.

r_{12}	r_{13}	r_{23}	b_2	b_3	R_{12}	R_{13}	B_{12}	B_{13}
$+0,606 \pm 0,069$	$+0,557 \pm 0,072$	$+0,264 \pm 0,083$	$+0,554$	$+2,691$	$+0,573$	$+0,518$	$+0,451$	$+2,065$
$+0,317 \pm 0,165$	$+0,173 \pm 0,171$	$-0,269 \pm 0,168$	$+0,369$	$+0,594$	$+0,384$	$+0,282$	$+0,457$	$+0,954$
$+0,555 \pm 0,065$	$+0,494 \pm 0,063$	$+0,237 \pm 0,076$	$+0,546$	$+2,342$	$+0,518$	$+0,448$	$+0,457$	$+1,816$

* Compare *L.M. II, 4*.

TABLE XI. Correlation between M , m , and μ_0 .

Group.	N	m_1	m_2	m_4	σ_1	σ_2	σ_4
α	132	m $+2,219 \pm 0,122$	m $+5,652 \pm 0,136$	m $-0,537 \pm 0,040$	m $1,407 \pm 0,087$	m $1,556 \pm 0,096$	m $0,464 \pm 0,029$
β	33	m $+2,776 \pm 0,307$	m $+5,939 \pm 0,263$	m $-0,245 \pm 0,050$	m $1,761 \pm 0,217$	m $1,512 \pm 0,186$	m $0,287 \pm 0,035$
$\alpha + \beta$	165	m $+2,330 \pm 0,117$	m $+5,709 \pm 0,121$	m $-0,479 \pm 0,035$	m $1,502 \pm 0,083$	m $1,553 \pm 0,085$	m $0,450 \pm 0,025$

TABLE XI. Contin.

r_{12}	r_{14}	r_{24}	b_2	b_4	R_{12}	R_{14}	B_{12}	B_{14}
$+0,598 \pm 0,070$	$+0,744 \pm 0,058$	$+0,360 \pm 0,081$	$+0,541$	$+2,255$	$+0,530$	$+0,707$	$+0,343$	$+1,843$
$+0,317 \pm 0,165$	$+0,209 \pm 0,170$	$-0,203 \pm 0,170$	$+0,369$	$+1,232$	$+0,375$	$+0,234$	$+0,436$	$+1,748$
$+0,533 \pm 0,066$	$+0,643 \pm 0,060$	$+0,290 \pm 0,075$	$+0,515$	$+2,145$	$+0,473$	$+0,602$	$+0,366$	$+1,779$

Here the indices 1, 2, 3, and 4 refer to the letters M , m , μ , and μ_0 respectively. N is the number of the stars, m , σ , r , and b the media, dispersions, correlation coefficients and regression coefficients respectively, and R , B the multiple correlation and regression coefficients respectively. All these quantities are computed by means of the formulæ given in the cited paper of CHARLIER, *L.M. I*, 66.

In this, like other statistical investigations, it proved necessary before making the numerical computation to reject certain stars on account of their possessing very large values of the variates. In this case I excluded a few stars whose values of M , m , μ , and μ_0 exceeded three times the corresponding dispersions.

By means of the tables above we can immediately write down the equations of regression of M on one or two of the other variates. These equations are:

From table X:

$$\text{Group } \alpha: \begin{cases} M - 2,145 = +0,554 (m - 5,590); \\ M - 2,145 = +2,691 (\mu - 0,405); \\ M - 2,145 = +0,451 (m - 5,590) + 2,065 (\mu - 0,405); \end{cases}$$

$$\text{Group } \beta: \begin{cases} M - 2,776 = +0,369 (m - 5,939); \\ M - 2,776 = +0,594 (\mu - 0,711); \\ M - 2,776 = +0,457 (m - 5,939) + 0,954 (\mu - 0,711); \end{cases}$$

$$\text{Groups } \alpha \text{ and } \beta: \begin{cases} M - 2,262 = +0,546 (m - 5,684); & (27) \\ M - 2,262 = +2,342 (\mu - 0,445); & (28) \\ M - 2,262 = +0,457 (m - 5,684) + 1,816 (\mu - 0,445); \end{cases}$$

From table XI:

$$\text{Group } \alpha: \begin{cases} M - 2,219 = + 0,541 (m - 5,652); \\ M - 2,219 = + 2,255 \mu_0 + 0,537; \\ M - 2,219 = + 0,343 (m - 5,652) + 1,843 (\mu_0 + 0,537); \end{cases} \quad (29)$$

$$\text{Group } \beta: \begin{cases} M - 2,776 = + 0,369 (m - 5,939); \\ M - 2,776 = + 1,282 (\mu_0 + 0,245); \\ M - 2,776 = + 0,436 (m - 5,939) + 1,748 (\mu_0 + 0,245); \end{cases} \quad (30)$$

$$\text{Groups } \alpha \text{ and } \beta: \begin{cases} M - 2,330 = + 0,515 (m - 5,709); \\ M - 2,330 = + 2,146 (\mu_0 + 0,479); \\ M - 2,330 = + 0,366 (m - 5,709) + 1,779 (\mu_0 + 0,479). \end{cases} \quad (31)$$

43. What conclusions can now be drawn from the numbers given above? In discussing results obtained from parallax stars we must always bear in mind that such stars are not random samples, on the contrary they make up a highly selected material. Parallax stars are firstly apparently very luminous stars, secondly of the apparently fainter stars attention has chiefly been paid to those with large proper motion, and both these properties make probable a relatively small distance from us. Thus the population of the parallax stars cannot be representative of the whole stellar system, and therefore the conclusion drawn from the results obtained from an examination of such stars cannot, in general, be thought of as valid for all stars of the system. On the contrary, they are only valid for the stars in our immediate neighbourhood. This I have remarked at the beginning of the discussion in order not to be obliged to repeat it at every conclusion.

The mean of the absolute magnitude is about $+ 2^m,3$ and its dispersion $+ 1^m,5$ these figures being obtained from all the stars. The stars of group β are on the average intrinsically fainter than those of group α , which fact can be explained by looking at the mean values of m and μ_0 (or μ) valid for these two groups. The stars of group β are on the average apparently fainter than those of group α , and further they are on the average nearer to us than these. For, as is seen from the tables X and XI, in group β about two thirds of the stars have values of μ_0 lying between the limits $- 0,532$ and $+ 0,042$, corresponding to values of μ equal to $0'',294$ and $1'',102$, in group α the analogous limits are for μ_0 : $- 1,001$ and $- 0,073$, and for μ : $0'',100$ and $0'',845$. Now both these facts bring about the difference observed between the mean of the absolute magnitudes as obtained from groups α and β . In the fact that stars of group β are in the mean nearer to us than those of group α , we can also trace the fact that in determining parallaxes by means of the »spectroscopical» method one is far more independent of the distances of the stars than when the usual »trigonometrical» method is applied. From the mean and dispersion of μ_0 (or μ) we also see the selection of very near lying stars for parallax determinations.

The mean of the apparent magnitude being $+ 5^m,7$ and its dispersion $+ 1^m,6$, we see that parallax stars are for the most part apparently faint. Their proper

motion, however, being on the average very great, we can conclude that they are mostly dwarf stars. However, as we shall see further on, there are some giants among them though they are rather few in number.

Looking now at the correlation coefficients, we find first that the correlation between M and m is rather strongly positive. If all stars were at the same distance from us, the correlation between M and m would, of course, be such as found here. But the distance varying within wide limits and being in most cases unknown to us, an apparently bright star may also be absolutely bright if it is at a large distance from us, or it may be absolutely faint if this distance is small. Thus we cannot a priori state anything about the correlation in question, and the value of it found here may be due to the selection of apparently faint stars with large proper motions. This selection of course also gives rise to the positive correlation between M and μ found in this investigation. Concerning the correlation between m and μ it would theoretically be negative. For the greater m is, the greater is also on the average the distance, and consequently the smaller on the average the proper motion. However, the stars in group α yield a positive value of this correlation which in itself is the mathematical expression of the selection of parallax stars often referred to above. It only shows that in order to determine the parallax of an apparently faint star one cannot hope to get a definite result unless the star in question has a large proper motion and thus probable is at a small distance from us.

44. The equations (27), (28), and (31), which represent the regressions of M on m , μ , and μ_0 respectively I have graphically represented in figures 1, 2, and 3, page 39, where the small circles give the values of M , derived from the correlation tables. From figure 1 it seems as if the regression of M on m would not be strictly linear, which would indicate that one or both of these two variates should not be normally distributed at least as concerns the parallax stars. Nor is the regression of M on μ linear, the reason of course being that the frequency curve of μ is of type B , whereas that of M is of type A , but introducing $\log \mu$ ($= \mu_0$) instead of μ , the regression of M on this new variate appears to be approximately linear.

The meaning of the regression equations being to give the most probable value of one variate (in this case M) for assigned values of the others I have made use of the equations (29) and (30) and by means of them recalculated the absolute magnitude for all the stars here employed. The mean of the absolute magnitudes thus obtained was $+2^m.261$, whereas the mean of the same quantities obtained from the parallaxes turned out to be $+2^m.265$.

Fig. 1.
Correlation between M and m .

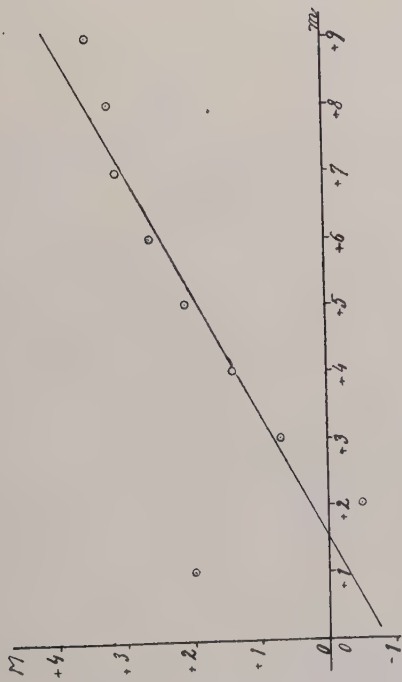


Fig. 2.
Correlation between M and μ .

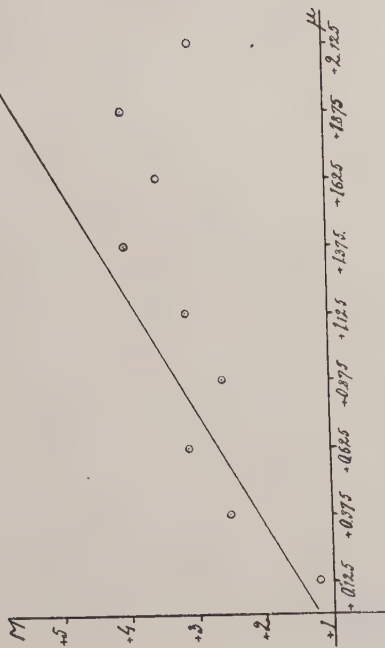


Fig. 3.
Correlation between M and $\log \mu$.

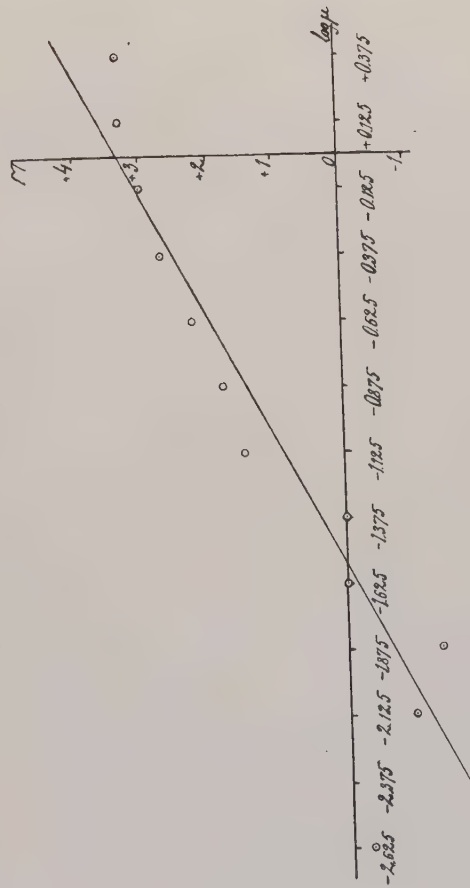
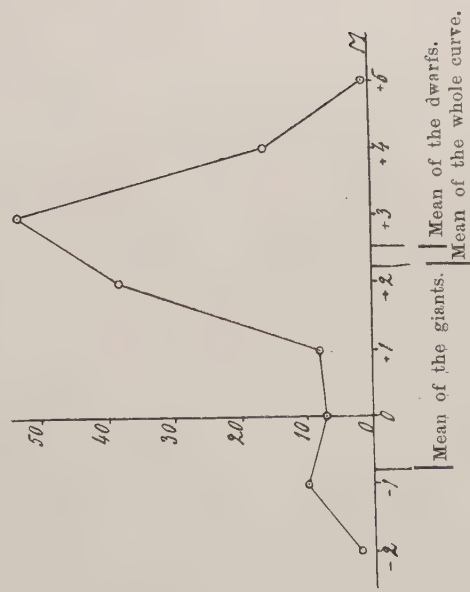


Fig. 4.
Frequency curve of M .



Mean of the dwarfs.
Mean of the whole curve.

TABLE XII. Frequency of absolute magnitude.

M between	Number of stars
$^m -2,5$ and $^m -1,5$	2
$^m -1,5$ „ $^m -0,5$	10
$^m -0,5$ „ $^m +0,5$	7
$^m +0,5$ „ $^m +1,5$	8
$^m +1,5$ „ $^m +2,5$	38
$^m +2,5$ „ $^m +3,5$	53
$^m +3,5$ „ $^m +4,5$	16
$^m +4,5$ „ $^m +5,5$	1
Sum: 135	

45. In figure 4 page 39 I have shown the frequency curve of the absolute magnitude of the stars contained in group α , i. e. ADAMS' catalogue. It is the graph of the following frequency table.

Both from the figure and the table it is seen that the frequency curve of M has two maxima. This was also observed by ADAMS*. I have, however, tried to dissect this frequency curve analytically into two normal components, thereby using the abridged method given by CHARLIER** which dispenses from the trouble of solving PEARSON'S nonic.

The characteristics of the above frequency curve necessary for the dissection being

$$\begin{aligned}\text{Mean} &= m_1 = + 2,207 \pm 0,123, \\ \text{Dispersion} &= \sigma_1 = 1,436 \pm 0,087, \\ \text{Dissymmetri} &= S_1 = 3\beta_3 = + 0,5361 \pm 0,1663, \\ \text{Excess} &= E_1 = 3\beta_4 = + 0,0861 \pm 0,0527, \\ &\beta_5 = - 0,0293,\end{aligned}$$

I obtained the result that the given curve is the sum of two normal component curves, one of which has its mean at $^m -0,81$ and the dispersion $0,75$ and including 15 per cent of the stars used, whereas the other curve includes the remaining 85 per cent of the stars and has its mean at $^m +2,51$ and a dispersion equal to $0,80$. The means of the component and resultant curves are indicated in figure 4, page 39, by the lines perpendicular to the base line. It may be mentioned here that RUSSEL found from his known diagrams in *Nature*, vol. 93, that the giants of the $F3$ -stars should have an absolute magnitude of about $^m 0,0$ and the dwarfs of the same type $^m +3,3$ ***.

Hence it is by this procedure established that the population of parallax stars of type F consists of two different kinds of stars, namely giant and dwarf stars. Whether this phenomenon is to be found in the whole mass of the F -stars is a question still open for discussion. In the case of the parallax stars it is, however, easy to show that it can be a consequence of the mode in which such stars are selected. Indeed, observing apparently luminous stars, it is obvious that among them there is one part with large and another part with small absolute luminosity. Hence we get some giants. The dwarfs are chiefly coming from the apparently faint stars with large proper motions.

* Ap. J. 46.

** L.M. II, 4.

*** M.N. 75, page 327.

46. In his lectures during the autumn term of 1918 CHARLIER has given a method for computing the star density in the vicinity of the sun *. It is based on the supposition that this density is constant, which may be regarded as approximately true if we confine us to stars brighter than the sixth magnitude. Its application requires the knowledge of

- 1) the value of the dispersion of the absolute magnitude,
- 2) the number of the stars whose absolute and apparent magnitude is smaller than 6 and whose distance from us is smaller than 1 Sir. (or parallax greater than $0''.206$), and
- 3) the number of the stars apparently brighter than the sixth magnitude.

These quantities being known we then get the sought density, D_0 , by solving a certain transcendental equation.

In paragraph 43 I have warned against generalizing the conclusions drawn from the numbers in the tables X and XI. However, as in chapter I I have deduced a value of D_0 , it might be of interest to see what value of it would be obtained by the above method, if the value of the dispersion of the absolute magnitude found in this chapter were employed.

This dispersion being equal to $1,5^m$ and the two other numbers necessary for computing the value of D_0 being 1 and 606 respectively, I got the number of R -stars per cubicsiriameter in the neighbourhood of the sun:

$$D_0 = 0,244.$$

Now this result, based as it is on only one star absolutely and apparently brighter than the sixth magnitude whose parallax is greater than $0''.206$, might be considered as rather untrustworthy, were it not that the star density here computed is of the same order of magnitude as that obtained in chapter I. I there got

$$D_0 = 0,124.$$

The difference between these two values of D_0 may partly be due to the non-constancy of R , partly to the uncertainty of the value of the dispersion of the absolute magnitude used here.

* Not yet published.

CHAPTER IV.

A method for determining the distribution in space of the stars.

47. Having computed the constant R by means of the method given in *L.M. I*, 77, the distance of every star is obtained from

$$r = R \cdot 10^{0,2 m}.$$

Thus we can get the coordinates of the stars in a certain fundamental system of coordinates, for instance the equatorial system, and, treating them statistically, we can determine the centre and the axes of our stellar system. This has been done by CHARLIER* for the B -stars. When I had determined the value of R for the F -stars (summer 1916) I intended to make an analogous computation for them. However, the value of R being rather small for these stars, 0,69, the greatest distance to which my material reached was

$$0,69 \cdot 10^{0,2 \cdot 5,99} = 10,89 \text{ Sir.}$$

As CHARLIER found from the B -stars the centre of the Milky Way at a distance from us of about 18 Sir. and the least axes of it equal to about 13 Sir., it was obvious that the F -stars brighter than the sixth magnitude would yield definite values neither of the coordinates of the centre nor of the axes of the stellar system.

Then I learned that at *Harvard College Observatory* a catalogue which in its complete state was to contain about 220000 stars of different spectral classes was in preparation. Hoping to get from this catalogue a material vast enough for a determination of the characteristics of the stellar system from the F -stars, I put off this computation, thinking to perform it when the catalogue was completed. As, however, only the first volume of it has hitherto been published, I have been obliged to give up this plan.

* *L.M. II*, 14.

48. The method used by CHARLIER is, however, somewhat lengthy and laborious, if the number of the stars is very great. I have therefore worked out a method which is numerically shorter. The application of it is founded on the following suppositions:

I. R (and consequently M) is approximately constant for the stars of a certain spectral type.

II. The apparent magnitude, m , is distributed according to the law

$$\varphi(m) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\sigma^2}},$$

where m_0 and σ are the mean and dispersion respectively of m .

III. Making use of the »squares», I suppose that every star has the same coordinates as the centre of the square in which it is situated.

Further are required the numerical values of R , and of m_0 and σ for all the stars of the spectral type considered, in every square separately.

We now have

$$\varphi(m)dm = \frac{dm}{\sigma\sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\sigma^2}} \quad (32)$$

and

$$r = R \cdot 10^{0.2m} = R e^{bm}, \quad (33)$$

where $b = \frac{0.2}{10 \log e} = 0.4605$. From (33) we get

$$dr = b \cdot r dm \therefore dm = \frac{dr}{r} \cdot \frac{1}{b}. \quad (34)$$

Inserting (33) and (34) in (32) and putting

$$\rho = \log R \quad (35)$$

we get

$$\varphi\left(\frac{1}{b}(\log r - \rho)\right) \frac{dr}{r} \cdot \frac{1}{b} = \frac{dr}{r b \sigma \sqrt{2\pi}} e^{-\frac{\left(\frac{\log r}{b} - \frac{\rho}{b} - m_0\right)^2}{2\sigma^2}}$$

or

$$\psi(r) = \frac{1}{b\sigma\sqrt{2\pi}} \frac{1}{r} e^{-\frac{[\log r - (\rho + bm_0)]^2}{2(b\sigma)^2}}. \quad (36)$$

Putting

$$\alpha = b\sigma; \quad \rho + bm_0 = r_0 \quad (37)$$

we get from (36)

$$\psi(r) = \frac{1}{\alpha\sqrt{2\pi}} \frac{1}{r} e^{-\frac{(\log r - r_0)^2}{2\alpha^2}} \quad (38)$$

This is the frequency function of the distances of the stars under the suppositions made.

49. We can now easily deduce the moments of the ϕ -function. Calling them (as taken about the origin) ρ'_s we have

$$\rho'_s = \frac{1}{\alpha\sqrt{2\pi}} \int_0^{\infty} r^s \frac{dr}{r} e^{-\frac{(\log r - r_0)^2}{2\alpha^2}} : \frac{1}{\alpha\sqrt{2\pi}} \int_0^{\infty} \frac{dr}{r} e^{-\frac{(\log r - r_0)^2}{2\alpha^2}}$$

or

$$\rho'_s = I_2 : I_1.$$

By means of the substitution

$$t = \log r$$

we find

$$I_2 = e^{\frac{s^2}{2}\alpha^2 + sr_0}$$

and of course

$$I_1 = 1.$$

Then

$$\rho'_s = e^{\frac{s^2}{2}\alpha^2 + sr_0}$$

or by means of (35) and (37)

$$\rho'_s = R^s \cdot e^{\frac{s^2}{2}b^2\sigma^2 + sbm_0},$$

or

$$\rho'_s = R^s \cdot 10^{0,2\frac{s^2}{2}b\sigma^2 + 0,2sm_0}. \quad (39)$$

Hence we have the moments of the distribution of r expressed in the moments m_0, σ of the distribution of m , and the constant R .

50. We now consider the stars of a certain spectral type in two diametrically situated squares, compute for each of these squares the values of m_0 and σ , which we call $m_{0,1}, \sigma_1$ and $m_{0,2}, \sigma_2$ respectively, and have thus the mean of the distances as computed from the one square:

$$\rho_1^{(1)} = R \cdot 10^{0,2\frac{b}{2}\sigma_1^2 + 0,2m_{0,1}},$$

and from the other:

$$\rho_1^{(2)} = R \cdot 10^{0,2\frac{b}{2}\sigma_2^2 + 0,2m_{0,2}}.$$

Letting now r be reckoned positive for the stars in the northern squares and nega-

tive for those in the southern squares, we evidently have the mean value of r as computed from the stars in a double-square:

$$\bar{r}_{12} = p_1 \rho_1^{(1)} - p_2 \rho_1^{(2)}, \quad (40)$$

where

$$p_1 = \frac{n_1}{n}, \quad p_2 = \frac{n_2}{n}, \quad n = n_1 + n_2,$$

and n_1, n_2 are the number of the stars in the two squares. Such an $\bar{r}$ -quantity we get for every double-square.

Denoting by $\gamma_{13}, \gamma_{23}, \gamma_{33}$, the direction cosines between the axes of the equatorial system and the line of sight to the centre of a square, on account of the supposition III, page 43, we get for every star in a certain square an equation of the form of

$$r = \gamma_{13}x + \gamma_{23}y + \gamma_{33}z.$$

Here x, y, z are the equatorial coordinates of the star and γ_{ij} are known functions of the right ascension and declination of the centre of the northern square*. By, taking the means of both membra of this equation for all stars in a double-square we find

$$\bar{r} = \gamma_{13}x_0 + \gamma_{23}y_0 + \gamma_{33}z_0.$$

Here $\bar{r}$ is known from equation (40) and the direction cosines from e. g. table I in *L.M. II, 13*. Having 24 equations of this kind we obtain by a least squares solution the coordinates x_0, y_0, z_0 of the centre of our stellar system.

51. In order to get the second order moments of x, y, z about their means x_0, y_0, z_0 , we proceed in the following manner. On account of supposition III, page 43, we have for a certain square the equations

$$\begin{aligned} x_{ij} &= \gamma_{13}^{(i)} r_{ij}, \\ y_{ij} &= \gamma_{23}^{(i)} r_{ij}, \\ z_{ij} &= \gamma_{33}^{(i)} r_{ij}; \end{aligned} \quad (41)$$

the indices i, j refer to the square and the star respectively; x, y, z have the same meaning as before, and r is the distance. Denoting by $M_j(\xi)$ the mean value of ξ for varying j , i. e. the mean value of ξ for all stars in a certain square, we get from (41)

$$\begin{aligned} M_j(x_{ij}) &= \gamma_{13}^{(i)} M_j(r_{ij}) = \gamma_{13}^{(i)} \rho_1^{(i)}, \\ M_j(y_{ij}) &= \gamma_{23}^{(i)} M_j(r_{ij}) = \gamma_{23}^{(i)} \rho_1^{(i)}, \\ M_j(z_{ij}) &= \gamma_{33}^{(i)} M_j(r_{ij}) = \gamma_{33}^{(i)} \rho_1^{(i)}. \end{aligned} \quad (42)$$

* See for instance *L.M. I, 77*.

In the same manner we have from the equations

$$\begin{aligned}x_{ij}^2 &= \gamma_{13}^{(i)^2} r_{ij}^2, \\y_{ij}^2 &= \gamma_{23}^{(i)^2} r_{ij}^2, \\z_{ij}^2 &= \gamma_{33}^{(i)^2} r_{ij}^2, \\y_{ij} z_{ij} &= \gamma_{23}^{(i)} \gamma_{33}^{(i)} r_{ij}^2, \\z_{ij} x_{ij} &= \gamma_{33}^{(i)} \gamma_{13}^{(i)} r_{ij}^2, \\x_{ij} y_{ij} &= \gamma_{13}^{(i)} \gamma_{23}^{(i)} r_{ij}^2,\end{aligned}$$

which immediately follow from (42), the relations

$$\begin{aligned}M_j(x_{ij}^2) &= \gamma_{13}^{(i)^2} M_j(r_{ij}^2) = \gamma_{13}^{(i)^2} \rho_2^{(i)'}, \\M_j(y_{ij}^2) &= \gamma_{23}^{(i)^2} \rho_2^{(i)'}, \\M_j(z_{ij}^2) &= \gamma_{33}^{(i)^2} \rho_2^{(i)'}, \\M_j(y_{ij} z_{ij}) &= \gamma_{23}^{(i)} \gamma_{33}^{(i)} \rho_2^{(i)'}, \\M_j(z_{ij} x_{ij}) &= \gamma_{33}^{(i)} \gamma_{13}^{(i)} \rho_2^{(i)'}, \\M_j(x_{ij} y_{ij}) &= \gamma_{13}^{(i)} \gamma_{23}^{(i)} \rho_2^{(i)}.\end{aligned}\tag{43}$$

However, from the stars in a certain square we get

$$\begin{aligned}\nu_{11}^{(i)} &= M_j[(x_{ij} - x_0)^2] = M_j(x_{ij}^2) - 2x_0 M_j(x_{ij}) + x_0^2, \\ \nu_{22}^{(i)} &= M_j[(y_{ij} - y_0)^2] = M_j(y_{ij}^2) - 2y_0 M_j(y_{ij}) + y_0^2, \\ \nu_{33}^{(i)} &= M_j[(z_{ij} - z_0)^2] = M_j(z_{ij}^2) - 2z_0 M_j(z_{ij}) + z_0^2, \\ \nu_{23}^{(i)} &= M_j[y_{ij} - y_0](z_{ij} - z_0) = M_j(y_{ij} z_{ij}) - y_0 M_j(z_{ij}) - z_0 M_j(y_{ij}) + y_0 z_0, \\ \nu_{31}^{(i)} &= M_j[(z_{ij} - z_0)(x_{ij} - x_0)] = M_j(z_{ij} x_{ij}) - z_0 M_j(x_{ij}) - x_0 M_j(z_{ij}) + z_0 x_0, \\ \nu_{12}^{(i)} &= M_j[(x_{ij} - x_0)(y_{ij} - y_0)] = M_j(x_{ij} y_{ij}) - x_0 M_j(y_{ij}) - y_0 M_j(x_{ij}) + x_0 y_0,\end{aligned}\tag{44}$$

where $\nu_{\alpha\beta}^{(i)}$, ($\alpha, \beta = 1, 2, 3$), are the second order moments of the coordinates x, y, z , about their means, as computed from the stars in a certain square. By means of (42) and (43), (44) may be written

$$\begin{aligned}\nu_{11}^{(i)} &= \gamma_{13}^{(i)^2} \rho_2^{(i)'} - 2x_0 \gamma_{13}^{(i)} \rho_1^{(i)'} + x_0^2, \\ \nu_{22}^{(i)} &= \gamma_{23}^{(i)^2} \rho_2^{(i)'} - 2y_0 \gamma_{23}^{(i)} \rho_1^{(i)'} + y_0^2, \\ \nu_{33}^{(i)} &= \gamma_{33}^{(i)^2} \rho_2^{(i)'} - 2z_0 \gamma_{33}^{(i)} \rho_1^{(i)'} + z_0^2, \\ \nu_{23}^{(i)} &= \gamma_{23}^{(i)} \gamma_{33}^{(i)} \rho_2^{(i)'} - (y_0 \gamma_{33}^{(i)} + z_0 \gamma_{23}^{(i)}) \rho_1^{(i)'} + y_0 z_0, \\ \nu_{31}^{(i)} &= \gamma_{33}^{(i)} \gamma_{13}^{(i)} \rho_2^{(i)'} - (z_0 \gamma_{13}^{(i)} + x_0 \gamma_{33}^{(i)}) \rho_1^{(i)'} + z_0 x_0, \\ \nu_{12}^{(i)} &= \gamma_{13}^{(i)} \gamma_{23}^{(i)} \rho_2^{(i)'} - (x_0 \gamma_{23}^{(i)} + y_0 \gamma_{13}^{(i)}) \rho_1^{(i)'} + x_0 y_0.\end{aligned}\tag{45}$$

The moments as computed from *all* squares are now

$$\nu_{\alpha\beta} = \frac{1}{N} \sum_{i=1}^{48} n_i \nu_{\alpha\beta}^{(i)}, \quad (\alpha, \beta = 1, 2, 3),\tag{46}$$

where n_i is the number of the stars in the square corresponding to the index i ,

and N is the total number of the stars, $N = \sum_{i=1}^{48} n_i$. From (45) and (46) we now get

$$\begin{aligned}
 v_{11} &= \frac{1}{N} [\Sigma n_i \gamma_{13}^{(i)2} \rho_2^{(i)'} - 2x_0 \Sigma n_i \gamma_{13}^{(i)} \rho_1^{(i)'}] + x_0^2, \\
 v_{22} &= \frac{1}{N} [\Sigma n_i \gamma_{23}^{(i)2} \rho_2^{(i)'} - 2y_0 \Sigma n_i \gamma_{23}^{(i)} \rho_1^{(i)'}] + y_0^2, \\
 v_{33} &= \frac{1}{N} [\Sigma n_i \gamma_{33}^{(i)2} \rho_2^{(i)'} - 2z_0 \Sigma n_i \gamma_{33}^{(i)} \rho_1^{(i)'}] + z_0^2, \\
 v_{23} &= \frac{1}{N} [\Sigma n_i \gamma_{23}^{(i)} \gamma_{33}^{(i)} \rho_2^{(i)'} - y_0 \Sigma n_i \gamma_{33}^{(i)} \rho_1^{(i)'} - z_0 \Sigma n_i \gamma_{23}^{(i)} \rho_1^{(i)'}] + y_0 z_0, \\
 v_{31} &= \frac{1}{N} [\Sigma n_i \gamma_{33}^{(i)} \gamma_{13}^{(i)} \rho_2^{(i)'} - z_0 \Sigma n_i \gamma_{13}^{(i)} \rho_1^{(i)'} - x_0 \Sigma n_i \gamma_{33}^{(i)} \rho_1^{(i)'}] + z_0 x_0, \\
 v_{12} &= \frac{1}{N} [\Sigma n_i \gamma_{13}^{(i)} \gamma_{23}^{(i)} \rho_2^{(i)'} - x_0 \Sigma n_i \gamma_{23}^{(i)} \rho_1^{(i)'} - y_0 \Sigma n_i \gamma_{13}^{(i)} \rho_1^{(i)'}] + x_0 y_0,
 \end{aligned} \tag{47}$$

where Σ stands for $\sum_{i=1}^{48}$. These equations give us the second order moments of the equatorial coordinates x, y, z about their means x_0, y_0, z_0 , when these latter have been obtained according to paragraph 50 and the $\rho_1^{(i)'}$, $\rho_2^{(i)'}$ have been computed by means of (39) from the values of m_0 and σ valid for the different squares.

52. The remaining task is now to determine the principal axes and their directions, of the ellipsoid defined by the moments in (47). The method is the usual one, and has been given for instance in *L.M. II, 14*.

53. As mentioned above I have not been able to apply this method on the *F*-stars, the present material being too scanty. However in order to test its adaptability I have by means of it redetermined the centre and the axes of the stellar system consisting of the *B3*- and *B5*-stars. These quantities have previously been given by CHARLIER*. The reason why I chose these very stars is that they have nearly the same value of R , viz. 3.4 Sir.*. Further practically all *B*-stars are cataloguized, and hence we can get the required quantities m_0 and σ for these stars. As however they are relatively few in number and strongly condensed towards the Milky Way there are certain squares mainly at the galactic poles which include only a small number of them. These squares I have not considered in the computation, as the values of m_0 and σ for them become too unreliable, and therefore my results do not coincide with CHARLIER's. In the small table below I have put together CHARLIER's results and my own.

* *L.M. II, 14*.

TABLE XIII. Centre and axial lengths of our stellar system.

Investigator	D (Sir)	α_0	δ_0	Σ_1			Σ_2			Σ_3		
				Length (Sir)	α	δ	Length (Sir)	α	δ	Length (Sir)	α	δ
CHARLIER (All B -stars)	18,2	$7,4^h$	$-55,6^0$	37,2	—	—	37,2	—	—	13,1	$0,3^h$	$-28,7^0$
LUNDAHL ($B3$, $B5$ -stars)	38,4	4,8	$-64,1$	37,8	$19,5^h$	$+58,6^0$	27,7	$16,7^h$	$-24,5^0$	23,2	23,3	$-18,4$

Here D is the distance from the sun to the centre; α_0, δ_0 are the coordinates of the centre; $\Sigma_1, \Sigma_2, \Sigma_3$ are the lengths of the principal axes of the stellar system.

Considering the bad determination of m_0 and σ in some of the squares the differences between the two sets of results are not unduly great, and it is to be expected that, if there were up to some hundreds of stars in every square, the results obtained by the two methods would be almost identical.

54. However, even when the *Harvard Catalogue* mentioned above has been finished we certainly shall not know *all* stars of a certain spectral type except the O - and B -types. In order then to get the values of m_0 and σ we must make use of some method giving the characteristics of a shortened normal frequency curve. Such a method has indeed been given by CHARLIER in *L.M. II*, 8. Its application postulates that we know up to what magnitude, m' say, our catalogue is complete. Counting then the number of stars in a square up to three arbitrary magnitudes lower than m' we can, by solving a transcendental equation, get the total number of the stars in that square and then m_0 and σ are deduced by linear equations. For details of this method I refer to the cited memoir.

Having brought this investigation to an end I will now express my sincere thanks to Professor Dr. C. V. L. CHARLIER who has always embraced my work with the warmest interest and given me much advice of the greatest value. Also to my friends, the assistants at the Lund Observatory Dr. W. GYLLENBERG, Dr. S. WICKSELL, and Lic. phil. G. MALMQUIST I am indebted for valuable suggestions.

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ERRATA:

Page 3, line 16 from above stands: fainter, read: brighter.

Page 7, lines 4 and 7 from below stands: table III, read: table II.

Page 8, lines 15 and 21 from below stands: table III, read: table II.

Page 9, line 3 from above stands: fainter, read: brighter.

Page 10, line 2 from below stands: fainter, read: brighter.

Page 13, line 6 from above, read: theoretical linear cross motion components and radial velocities.

Page 15, line 10 from below, read: The left membrum of this equation should strictly be corrected for a certain constant, K, etc.